# Lectures on Macroeconomics

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## 2

# Consumption and Investment: Basic Infinite Horizon Models

In this chapter and the next we focus on the fundamentals of consumption and capital accumulation in dynamic nonmonetary equilibrium models. We introduce basic models—in this chapter, the Ramsey infinite horizon optimizing model, and in the next, overlapping generations models with finite horizon maximizers—and begin to analyze economic issues such as how much interest rates affect savings and whether the choice between tax and deficit financing affects capital accumulation.

Individuals are assumed in this chapter to have an infinite horizon, or to live forever.[1] The infinite horizon assumption turns out to have strong implications: together with the assumptions of competitive markets, constant returns to scale in production, and homogeneous agents, it typically implies that the allocation of resources achieved by a decentralized economy will be the same as that chosen by a central planner who maximizes the utility of the representative economic agent in the model. We demonstrate here the equivalence between the allocation of resources in the decentralized economy and in a planned economy.

We start this chapter by developing the Ramsey (1928) analysis of optimal economic growth under certainty, by deriving the intertemporal conditions that are satisfied on the optimal path that would be chosen by a central planner. We then show, in section 2.2, the equivalence of the optimal path to the equilibrium path of the decentralized economy. In section 2.3 we examine the effects of both lump-sum and capital taxation on the rate of saving and the equilibrium interest rate in the framework of the decentralized infinite horizon economy.

In section 2.4 we analyze the economy of a small country, showing how the evolution of the current account is determined by investment and saving behavior. We examine the response of the economy to supply shocks, showing under what circumstances a country will respond to an adverse shock by borrowing abroad.

In the final section, section 2.5, we discuss some of the special features and implications of the intertemporally separable utility function with constant rate of time preference used in the chapter, and examine alternative formulations.

## 2.1 The Ramsey Problem

Frank Ramsey[2] posed the question of how much a nation should save and solved it using a model that is now the prototype for studying the optimal intertemporal allocation of resources. The model presented in this section is essentially that of Ramsey.

The population, $N_t$, grows at rate $n$; it can be thought of as a family, or many identical families, growing over time. The labor force is equal to the population, with labor supplied inelastically. Output is produced using capital, $K$, and labor. There is no productivity growth.

The output is either consumed or invested, that is, added to the capital stock. Formally,

$$Y_t = F(K_t, N_t) = C_t + \frac{dK_t}{dt}. \tag{1}$$

For simplicity, we assume that there is no physical depreciation of capital, or that $Y_t$ is net rather than gross output.[3] The production function is homogeneous of degree one: that is, there are constant returns to scale.

In per capita terms

$$f(k_t) = c_t + \frac{dk_t}{dt} + nk_t, \tag{2}$$

where lowercase letters denote per capita (equal to per worker) values of variables so that $k$ is the capital-labor ratio and $f(k_t) \equiv F(K_t/N_t, 1)$; we assume $f(\cdot)$ to be strictly concave and to satisfy the following conditions, known as Inada conditions:

$$f(0) = 0, \qquad f'(0) = \infty, \qquad f'(\infty) = 0.$$

We also assume that the economy starts with some capital so that it can get production off the ground:

$$k_0 > 0.$$

The preferences of the family for consumption over time are represented by the utility integral:

$$U_s = \int_s^{\infty} u(c_t) \exp[-\theta(t - s)]\, dt. \tag{3}$$

The family's welfare at time $s$, $U_s$, is the discounted sum of instantaneous utilities $u(c_t)$. The function $u(\ )$ is known as the instantaneous utility function, or as "felicity"; $u(\ )$ is nonnegative and a concave increasing function of the per capita consumption of family members. The parameter $\theta$ is the rate of time preference, or the subjective discount rate, which is assumed to be strictly positive.[4]

### The Command Optimum

Suppose that a central planner wants at time $t = 0$ to maximize family welfare. The only choice that has to be made at each moment of time is how much the representative family should consume and how much it should add to the capital stock to provide consumption in the future. The planner has to find the solution to the following problem:

$$\max U_0 = \int_0^{\infty} u(c_t) \exp(-\theta t)\, dt \tag{4}$$

subject to (2) and the constraints

$k_0$ given; $\qquad k_t, c_t \geqslant 0$ for all $t$.

We characterize the solution using the maximum principle.[5] The optimal solution is obtained by setting up the present value Hamiltonian function:

$$H_t = u(c_t) \exp(-\theta t) + \mu_t[f(k_t) - nk_t - c_t]. \tag{5}$$

The variable $\mu$ is called the *costate* variable associated with the *state* variable $k$; equivalently it is the multiplier on the constraint (2). The value of $\mu_t$ is the marginal value as of time zero of an additional unit of capital at time $t$.

It is often more convenient to work, instead, with the marginal value, as of time $t$, of an additional unit of capital at time $t$, $\lambda_t \equiv \mu_t \exp(\theta t)$; we shall do so here. Replacing $\mu_t$ by $\lambda_t$ in (5) gives

$$H_t = [u(c_t) + \lambda_t(f(k_t) - nk_t - c_t)] \exp(-\theta t). \tag{5'}$$

We do not explicitly impose the nonnegativity constraints on $k$ and $c$.

Necessary and sufficient conditions for a path to be optimal under the assumptions on the utility and production functions made here are that[6]

$$H_c = 0,$$

$$\frac{d\mu_t}{dt} = -H_k,$$

$$\lim_{t\to\infty} k_t\mu_t = 0.$$

Using the definition of $H(\cdot)$ and replacing $\mu$ by $\lambda$, we get

$$u'(c_t) = \lambda_t, \tag{6}$$

$$\frac{d\lambda_t}{dt} = \lambda_t[\theta + n - f'(k_t)], \tag{7}$$

$$\lim_{t\to\infty} k_t u'(c_t) \exp(-\theta t) = 0. \tag{8}$$

Equations (6) and (7) can be consolidated to remove the costate variable $\lambda$, yielding

$$\frac{du'(c_t)/dt}{u'(c_t)} = \theta + n - f'(k_t), \tag{7'}$$

or equivalently

$$\left[\frac{c_t u''(c_t)}{u'(c_t)}\right]\left(\frac{dc_t/dt}{c_t}\right) = \theta + n - f'(k_t).$$

The expression $cu''(c)/u'(c)$ will recur often in this book. It reflects the curvature of the utility function. More precisely, it is equal to the elasticity of marginal utility with respect to consumption. If utility is nearly linear and if marginal utility is nearly constant, then the elasticity is close to zero. This elasticity is itself closely related to the *instantaneous elasticity of substitution.* The elasticity of substitution between consumption at two points in time, $t$ and $s$, is given by

$$\sigma(c_t) \equiv -\frac{u'(c_s)/u'(c_t)}{c_s/c_t}\,\frac{d(c_s/c_t)}{d[u'(c_s)/u'(c_t)]}.$$

Taking the limit of that expression as $s$ converges to $t$ gives $\sigma = -u'(c_t)/u''(c_t)c_t$ so that $\sigma(c_t)$ is the inverse of the negative of the elasticity of marginal utility. When utility is nearly linear, the elasticity of substitution is very large. Using the definition of $\sigma$, (7′) can be rewritten as

$$\frac{dc_t/dt}{c_t} = \sigma(c_t)[f'(k_t) - \theta - n]. \tag{7''}$$

The key conditions are (7) [or (7′) or (7″)] and (8). Equation (7) is the Euler equation, the differential equation describing a necessary condition that has to be satisfied on any optimal path. It is the continuous time analogue of the standard efficiency condition that the marginal rate of substitution be equal to the marginal rate of transformation, as we shall show shortly. The condition is also known as the Keynes-Ramsey rule. It was derived by Ramsey in his classic article, which includes a verbal explanation attributed to Keynes. We now develop an intuitive explanation of this repeatedly used condition.

## The Keynes-Ramsey Rule

The easiest way to understand the Keynes-Ramsey rule is to think of time as being discrete and to consider the choice of the central planner in allocating consumption between time $t$ and $t + 1$. If he decreases consumption at time $t$ by $dc_t$, the loss in utility at time $t$ is equal to $u'(c_t)\,dc_t$. This decrease in consumption at time $t$, however, allows for more accumulation and thus more consumption at time $t + 1$: consumption per capita can be increased by $(1 + n)^{-1}[1 + f'(k_t)]\,dc_t$, leading to an increase in utility at $t + 1$ of $(1 + n)^{-1}[1 + f'(k_t)]u'(c_{t+1})\,dc_t$. Along the optimal path small reallocations in consumption must leave welfare unchanged so that the loss in utility at time $t$ must be equal to the discounted increase in utility at time $t + 1$. Thus

$$u'(c_t) = (1 + \theta)^{-1}(1 + n)^{-1}[1 + f'(k_t)]u'(c_{t+1}).$$

This condition can be rewritten as

$$\frac{(1 + \theta)^{-1}u'(c_{t+1})}{u'(c_t)} = \frac{1 + n}{1 + f'(k_t)}, \tag{9}$$

which states that the marginal rate of substitution (MRS) between consumption at times $t$ and $t + 1$ is equal to the marginal rate of transformation (MRT), from production, between consumption at times $t$ and $t + 1$. If the period is short enough, this condition reduces to equation (7′).

A more rigorous argument runs as follows: Consider two points in time, $t$ and $s$, $s > t$. We now imagine reallocating consumption from a small interval following $t$ to an interval of the same length following $s$. Decrease $c_t$ by amount $\Delta c_t$ at time $t$ for a period of length $\Delta t$, thus increasing capital accumulation by $\Delta c_t \Delta t$. That capital is allowed to accumulate between $t + \Delta t$ and $s$, with consumption over that interval unchanged from its

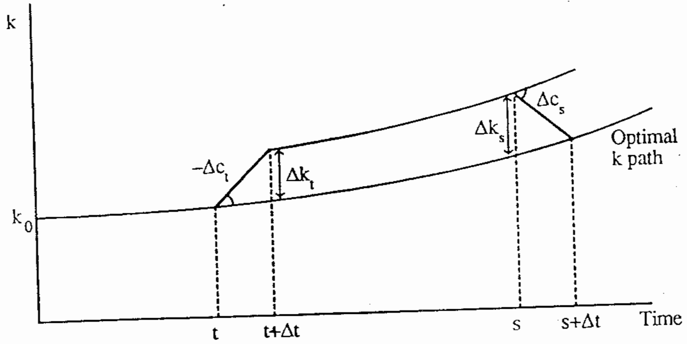

**Figure 2.1**
The Keynes-Ramsey rule

original value. All the increased capital is consumed during an interval of length $\Delta t$ starting at $s$, with consumption thereafter being unchanged from the level on the original path. This variation from the optimal path is illustrated in figure 2.1.

For sufficiently small $\Delta c$ and $\Delta t$, such a reallocation should have no effect on welfare, provided the path is optimal. Thus

$$u'(c_t)\Delta c_t \Delta t + u'(c_s)\exp[-\theta(s-t)]\Delta c_s \Delta t = 0.$$

The relation between $\Delta c_t$ and $\Delta c_s$ is implied by

$$\Delta c_t \Delta t = \Delta k_t, \qquad \Delta c_s \Delta t = \Delta k_s,$$

and

$$\Delta k_s = -\Delta k_t \exp\left\{\int_{t+\Delta t}^{s} [f'(k_v) - n]\, dv\right\}.$$

Capital accumulated in the first interval $\Delta t$ grows at the rate $f'(k) - n$ between $t + \Delta t$ and $s$.

Eliminating $\Delta c$'s and $\Delta k$'s from the preceding relations gives

$$\frac{u'(c_t)}{u'(c_s)\exp[-\theta(s-t)]} = \exp\left\{\int_{t+\Delta t}^{s} [f'(k_v) - n]\, dv\right\}. \tag{10}$$

Equation (10) has the same interpretation as equation (9), namely, that marginal rates of substitution and transformation are equal.

As this equality must hold for all $t$ and $s$, it follows that

$$\lim_{s\to t} \frac{d\mathrm{MRS}(t, s)}{ds} = \lim_{s\to t} \frac{d\mathrm{MRT}(t, s)}{ds}.$$

Applying this to (10) gives equation (7′).

The Keynes-Ramsey rule, in discrete or continuous time, implies that consumption increases, remains constant, or decreases depending on whether the marginal product of capital (net of population growth) exceeds, is equal to, or is less than the rate of time preference. This rule is quite fundamental and quite intuitive: the higher the marginal product of capital relative to the rate of time preference, the more it pays to depress the current level of consumption in order to enjoy higher consumption later. Thus, if initially the marginal product of capital is high, consumption will be increasing over time on the optimal path. Equation (7″) shows the specific role of the elasticity of substitution in this condition: the larger this elasticity, the easier it is, in terms of utility, to forgo current consumption in order to increase consumption later, and thus the larger the rate of change of consumption for a given value of the excess of the marginal product over the subjective discount rate.

### The Transversality Condition

Equation (8), the transversality condition, is best understood by considering the same maximization problem with the infinite horizon replaced by a finite horizon $T$. In this case, if $u'(c_T)\exp(-\theta T)$ were positive (i.e., if the present value of the marginal utility of terminal consumption were positive), it would not be optimal to end up at time $T$ with a positive capital stock because it could, instead, be consumed.[7] The condition would be

$$k_T u'(c_T)\exp(-\theta T) = 0.$$

The infinite horizon transversality condition (TVC) can be thought of as the limit of this condition as $T$ becomes large.[8]

### Two Useful Special Cases

#### *CRRA*

Two instantaneous utility functions are frequently used in intertemporal optimizing models. The first is the constant elasticity of substitution, or isoelastic, function:[9]

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \qquad \text{for } \gamma > 0, \gamma \neq 1,$$
$$= \ln c, \qquad \text{for } \gamma = 1.$$

The basic economic property of this function is implied by its name. The elasticity of substitution between consumption at any two points in time, $t$ and $s$, is constant and equal to $(1/\gamma)$. Thus, in equation (7″), $\sigma$ is no longer a function of consumption. The elasticity of marginal utility is equal to $-\gamma$.

When this instantaneous utility function is used to describe attitudes toward risk, something we shall do later in the book when we allow for uncertainty, $\gamma$ has an alternative interpretation. It is then also the coefficient of relative risk aversion, defined as $-u''(c)c/u'(c)$. Thus this function is also called the constant relative risk aversion (CRRA) utility function.[10]

Substantial empirical work has been devoted to estimating $\sigma$ under the assumption that it is indeed constant, by looking at how willing consumers are to shift consumption across time in response to changes in interest rates. Estimates of $\sigma$ vary substantially but usually lie around or below unity: the bulk of the empirical evidence suggests a relatively low value of the elasticity of substitution.

*CARA*

The second often used class of utility functions is the exponential, or constant absolute risk aversion (CARA), of the form

$$u(c) = -\left(\frac{1}{\alpha}\right)\exp(-\alpha c), \qquad \alpha > 0.$$

Under this specification the elasticity of marginal utility is equal to $-\alpha c$, and the instantaneous elasticity of substitution is equal to $(\alpha c)^{-1}$; thus $\sigma$ is decreasing in the level of consumption.

When interpreted as describing attitudes toward risk, this function implies constant absolute risk aversion, with $\alpha$ being the coefficient of absolute risk aversion, $-u''(c)/u'(c)$. Constant absolute risk aversion is usually thought of as a less plausible description of risk aversion than constant relative risk aversion; the CARA specification is, however, sometimes analytically more convenient than the CRRA specification and thus also belongs to the standard tool kit.

For the CARA utility function, the Euler equation becomes

$$\frac{dc}{dt} = \alpha^{-1}[f'(k) - n - \theta]. \qquad (7''')$$

In this case the change in consumption is proportional to the excess of the marginal product of capital (net of population growth) over the discount rate.

## Steady State and Dynamics

The optimal path is characterized by equations (7′), (8), and the constraint (2). We start with the steady state. In steady state both the per capita capital stock, $k$, and the level of consumption per capita, $c$, are constant. We denote the steady state values of these variables by $k^*$ and $c^*$, respectively.

### *The Modified Golden Rule*

From (7), with $dc/dt$ equal to zero, we have the modified golden rule relationship:

$$f'(k^*) = \theta + n. \qquad (11)$$

The marginal product of capital in steady state is equal to the sum of the rate of time preference and the growth rate of population. Corresponding to the optimal capital stock $k^*$ is the steady state level of consumption, implied by (2):

$$c^* = f(k^*) - nk^*. \qquad (12)$$

The *golden rule* itself is the condition $f'(k) = n$: this is the condition on the capital stock that maximizes *steady state* consumption per capita.[11] The modification in (11) is that the capital stock is reduced below the golden rule level by an amount that depends on the rate of time preference. Even though society or the family could consume more in a steady state with the golden rule capital stock, the impatience reflected in the rate of time preference means that it is not optimal to reduce current consumption in order to reach the higher golden rule consumption level.

The *modified golden rule* condition is a very powerful one: it implies that ultimately the productivity of capital, and thus the real interest rate,[12] is determined by the rate of time preference and $n$. Tastes and population growth determine the real interest rate $(\theta + n)$, and technology then determines the capital stock and level of consumption consistent with that interest rate.[13] Later in the chapter we will explore the sensitivity of the modified golden rule result to the formulation of the utility function $u(\cdot)$ in (1).

### *Dynamics*

To study dynamics, we use the phase diagram in figure 2.2, drawn in $(k, c)$ space.[14] All points in the positive orthant are feasible, except for points on

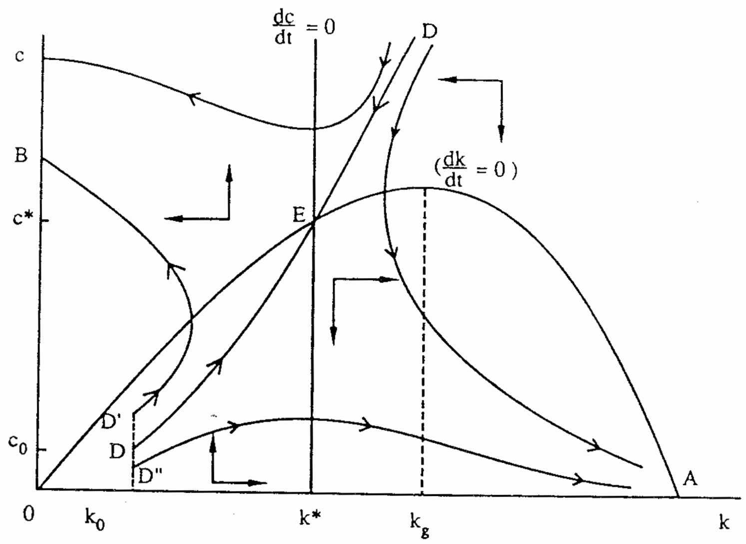

**Figure 2.2**
The dynamics of capital and consumption

the vertical axis above the origin: without capital (i.e., if $k = 0$), output is zero, and thus positive $c$ is not feasible.

The locus $dk/dt = 0$ starts from the origin, reaches a maximum at the golden rule capital stock $k_g$ at which $f'(k_g) = n$, and crosses the horizontal axis at point $A$ where $f(k) = nk$. The $dc/dt = 0$ locus is, from (7′), vertical at the modified golden rule capital stock, $k^*$.

Anywhere above the $dk/dt = 0$ locus, the capital–labor ratio $k$ is decreasing: consumption is above the level that would just maintain $k$ constant (i.e., the level of $c$ on the $dk/dt = 0$ curve.) Similarly, $k$ is increasing at points below the $dk/dt = 0$ locus. In the case of the $dc/dt = 0$ locus, consumption is increasing to the left of the locus, where $f'(k) > \theta + n$, and decreasing to the right of the locus. The vertical arrows demonstrate these directions of motion.[15]

There are three equilibria, the origin, if $\sigma^{-1}(0)$ is different from zero (see note 15), point $E$, and point $A$. In appendix A we show that only the trajectory $DD$, the *saddle point path*, that converges to $E$ satisfies the necessary conditions (2), (7′), and (8). On all other paths, either the Keynes-Ramsey condition eventually fails or the transversality condition is not satisfied.[16]

The central planner's solution to the optimizing problem (1) is fully summarized by the path $DD$. For each initial capital stock, this implies a

unique initial level of consumption. For instance, with initial capital stock $k_0$, the optimal initial level of consumption is $c_0$. Convergence of $c$ and $k$ to $c^*$ and $k^*$ is monotonic. Note that in this certainty model the central planner knows at time 0 what the level of consumption and the capital stock will be at every moment in the future.

## Local Behavior around the Steady State

Linearization of the dynamic system (2) and (7′) yields further insights into the dynamic behavior of the economy. Linearizing both equations in the neighborhood of the steady state gives

$$\frac{dc}{dt} = -\beta(k - k^*), \qquad \beta \equiv [-f''(k^*)c^*]\sigma(c^*) > 0, \tag{13}$$

and

$$\begin{aligned} \frac{dk}{dt} &= [f'(k^*) - n](k - k^*) - (c - c^*) \\ &= \theta(k - k^*) - (c - c^*). \end{aligned} \tag{14}$$

The solution to this system of linear differential equations is most easily found by reducing it to a single second-order equation in $k$. Differentiating (14) with respect to time, and using (13) to substitute for $dc/dt$, gives

$$\frac{d^2k}{dt^2} - \theta\left(\frac{dk}{dt}\right) - \beta k = -\beta k^*. \tag{15}$$

The roots of the characteristic equation associated with the second-order differential equation are $\theta \pm (\sqrt{\theta^2 + 4\beta})/2$. One root is positive and the other negative, implying the saddle point property: the presence of a positive root implies that for arbitrary initial conditions, the system explodes; for any given value of $k_0$, there is a unique value of $dk/dt$ such that the system converges to the steady state (see appendix B).

Let $\lambda$ be the negative, stable root. The solution for $k_t$ such that, starting from $k_0$, the system converges to $k^*$ is

$$k_t = k^* + (k_0 - k^*)\exp(\lambda t).$$

The speed of convergence is thus given by $|\lambda|$. In turn $|\lambda|$ is an increasing function of $f''$ and of $\sigma$, and a decreasing function of $\theta$. The higher the elasticity of substitution, the more willing people are to accept low consumption early on in exchange for higher consumption later and the faster capital accumulates and the economy converges to the steady state.[17]

## 2.2 The Decentralized Economy

Suppose that the economy is decentralized rather than centrally planned. There are two factor markets, one for labor and one for capital services. The rental price of labor, the wage, is denoted $w_t$; $r_t$ is the rental price of capital. There is a debt market in which families can borrow and lend.

There are many identical families, each with a welfare function given by equation (3). Each family decides, at any point in time, how much labor and capital to rent to firms and how much to save or consume. They can save by either accumulating capital or lending to other families. Families are indifferent as to the composition of their wealth, so the interest rate on debt must be equal to the rental rate on capital.[18]

There are many identical firms, each with the same technology as described by equation (2); firms rent the services of capital and labor to produce output.[19] The constant returns assumption means that the number of firms is of no consequence, provided the firms behave competitively, taking the prices (the real wage and rental rate on capital) facing them as given.[20]

Both families and firms have perfect foresight; that is, they know both current and future values of $w$ and $r$ and take them as given. (Under certainty, perfect foresight is the equivalent of rational expectations, an assumption we will discuss at length later.) More formally, let $\{w_t, r_t\}$, $t = [0, \infty)$, be the sequence of wages and rental rates. Then, given this sequence, each family maximizes at any time $s$

$$U_s = \int_s^\infty u(c_t) \exp[-\theta(t - s)]\, dt$$

subject to the budget constraint,

$$c_t + \frac{da_t}{dt} + na_t = w_t + r_t a_t, \qquad \text{for all } t,\ k_0 \text{ given}, \tag{16}$$

where

$$a_t \equiv k_t - b_{pt}.$$

Family wealth, or more precisely nonhuman wealth, is given by $a_t$, which is equal to holdings of capital, $k_t$, minus family debt, $b_{pt}$.

At any time $t$, the family supplies both capital and labor services inelastically: capital is the result of previous decisions and is given at time $t$; by assumption, labor is supplied inelastically. Thus the only decision the family has to make at each point in time is how much to consume or save.

Firms in turn maximize profits at each point in time. Since their technology is characterized by the production function (2), first-order conditions for profit maximization imply that

$$f'(k_t) = r_t, \qquad f(k_t) - k_t f'(k_t) = w_t. \tag{17}$$

Consider an arbitrary path of wages and rental rates. This sequence will lead each family to choose a path of consumption and wealth accumulation. Given that private debt must always be equal to zero in the aggregate, wealth accumulation will determine capital accumulation. The path of capital will in turn imply a path of wages and rental rates. The equilibrium paths of wages and rental rates are defined as those paths that reproduce themselves given optimal decisions by firms and households. We now characterize the equilibrium path of the economy.[21]

### The No-Ponzi-Game Condition

In stating the maximization problem of a family, we have not imposed the constraint that family nonhuman wealth, which is given by $a_t$ at time $t$, be nonnegative. In the absence of any restrictions on borrowing, the solution to the maximization problem is then a trivial one. It is for the family to borrow sufficiently to maintain a level of consumption such that the marginal utility of consumption equals zero (or an infinite level of consumption if marginal utility is always positive) and to let the dynamic budget constraint determine the dynamic behavior of $a$. From the budget constraint it follows that this path of consumption will lead to higher and higher levels of borrowing (negative $a$), borrowing being used to meet interest payments on the existing debt. Ultimately, net indebtedness per family member will be growing at rate $r_t - n$.

We need therefore an additional condition that prevents families from choosing such a path, with an exploding debt relative to the size of the family. At the same time we do not want to impose a condition that rules out temporary indebtedness.[22] A natural condition is to require that family debt not increase asymptotically faster than the interest rate:

$$\lim_{t\to\infty} a_t \exp\left[-\int_0^t (r_v - n)\, dv\right] \geqslant 0. \tag{18}$$

This condition is sometimes known as a no-Ponzi-game (NPG) condition.[23] Although (18) is stated as an inequality, it is clear that as long as marginal utility is positive, families will not want to have increasing wealth forever at rate $r - n$, and that the condition will hold as an equality. Thus in what follows we use the condition directly as an equality.

To see what the condition implies, let us first integrate the budget constraint from time 0 to some time $T$. This gives

$$\int_0^T c_t \exp\left[\int_t^T (r_v - n)\, dv\right] dt + a_T$$
$$= \int_0^T w_t \exp\left[\int_t^T (r_v - n)\, dv\right] dt + a_0 \exp\left[\int_0^T (r_v - n)\, dv\right].$$

Multiplying both sides by $\exp[-\int_0^T (r_v - n)\, dv]$, that is, discounting to time zero, letting $T$ go to $\infty$, and using the NPG condition, gives

$$\int_0^\infty c_t \exp\left[-\int_0^t (r_v - n)\, dv\right] dt = a_0 + h_0,$$

where

$$h_0 \equiv \int_0^\infty w_t \exp\left[-\int_0^t (r_v - n)\, dv\right] dt.$$

This condition implies that the present value of consumption is equal to total wealth, which is the sum of nonhuman wealth, $a_0$, and of human wealth, $h_0$, the present value of labor income. Thus condition (18) allows us to go from the dynamic budget constraint (16) to an intertemporal budget constraint.[24]

## The Decentralized Equilibrium

Maximization of (3) subject to (16) and (18), carried out by setting up a Hamiltonian, implies the following necessary and sufficient conditions:

$$\frac{du'(c_t)/dt}{u'(c_t)} = \theta + n - r_t, \tag{19}$$

$$\lim_{t\to\infty} a_t u'(c_t) \exp(-\theta t) = 0. \tag{20}$$

In equilibrium, aggregate private debt $b_{pt}$ must always be equal to zero: though each family assumes it can freely borrow and lend, in equilibrium there is neither lending nor borrowing. Thus $a_t = k_t$. Using this and equations (17) for $w_t$ and $r_t$ and replacing in (16) and (19) gives

$$c_t + \frac{dk_t}{dt} + nk_t = f(k_t), \tag{21}$$

$$\frac{du'(c_t)/dt}{u'(c_t)} = \theta + n - f'(k_t). \tag{22}$$

Equations (20), (21), and (22) characterize the behavior of the decentralized economy. Note that they are identical to equations (8), (2), and (7′) which characterize the behavior of the economy as chosen by a central planner. Thus the dynamic behavior of the decentralized economy will be the same as that of the centrally planned one. Our analysis of dynamics carries over to the decentralized economy.[25]

## The Role of Expectations

Equation (19), the Euler equation, gives the rate of change of consumption as a function of variables known at the current moment. It could be interpreted as suggesting that households need not form expectations of future variables in making their consumption/saving decisions and that the assumption of perfect foresight is not necessary. However, it is clear from the intertemporal budget constraint that the household cannot plan without knowing the entire path of both the wage and the interest rate. Expectations thus are crucial to the allocation of resources in the decentralized economy. In terms of the Euler equation, equation (19) only determines the rate of change, not the level of consumption.

Although it is difficult in general to solve explicitly for the level of consumption, this can be done easily when the utility function is of the CRRA family. In this case equation (19) gives

$$\frac{dc_t/dt}{c_t} = \sigma(r_t - n - \theta).$$

For a given value of initial consumption $c_0$, we can integrate this equation forward to get

$$c_t = c_0 \exp\left[\int_0^t \sigma(r_v - n - \theta)\, dv\right].$$

Replacing in the intertemporal budget constraint gives the value of $c_0$ consistent with the Euler equation and the budget constraint

$$c_0 = \beta_0(a_0 + h_0),$$

where

$$\beta_0^{-1} \equiv \left[\int_0^\infty \exp\left\{\int_0^t [(\sigma - 1)(r_v - n) - \theta\sigma]\, dv\right\} dt\right].$$

Consumption is a linear function of wealth, human and nonhuman. The parameter $\beta_0$ is the propensity to consume out of wealth. It is generally a function of the expected path of interest rates. An increase in interest rates, given wealth, has two effects. The first is to make consumption later more attractive: this is the substitution effect. The second is to allow for higher consumption now and later: this is the income effect. In general, the net effect on the marginal propensity to consume is ambiguous. For the logarithmic utility function, however, $\sigma = 1$, and the two effects cancel; the propensity to consume is then exactly equal to the rate of time preference, $\theta$, and is independent of the path of interest rates.

In general, expectations of interest rates affect both the marginal propensity to consume out of wealth and the value of wealth itself, through $h_0$. Expectations of wages also affect $c_0$ through $h_0$. Given these expectations, families decide how much to consume and save. This in turn determines capital accumulation and the sequence of factor prices.

What happens if expectations are incorrect? Agents will choose a different plan from our hypothetical central planner. When the divergence between actual and expected events causes them to revise their expectations, they will choose a new path that is optimal given their expectations. To pursue this line, we would have to specify how expectations are formed and revised. We defer that for later treatment.

## 2.3 The Government in the Decentralized Economy

In this section we introduce the government into the model. We assume that the government's spending requirements are fixed exogenously,[26] and we examine the effects on the economy's equilibrium of, first, changes in the level of government spending and, second, different ways of financing a given level of government spending—either through taxation or borrowing.

### Balanced Budget Changes in Government Spending

Suppose that a government is consuming resources and paying for them with taxes. The government's per capita demand for resources $g_t$ is exogenous and, further, does not directly affect the marginal utility of consumption of the representative household.[27] To begin with, let the government levy per capita lump-sum taxes $\tau_t = g_t$, so that the government budget is balanced at every moment.

The household flow budget constraint now becomes

$$c_t + \frac{da_t}{dt} + na_t = w_t + r_t a_t - \tau_t, \qquad a_t = k_t - b_{pt},$$

which, using the NPG condition, integrates to

$$\int_0^\infty c_t R_t\, dt = k_0 - b_{p0} + \int_0^\infty w_t R_t\, dt - \int_0^\infty \tau_t R_t\, dt,$$

or equivalently,

$$\int_0^\infty c_t R_t\, dt = k_0 - b_{p0} + h_0 - G_0, \tag{23}$$

where $R_t = \exp[-\int_0^t (r_v - n)\, dv]$ is the factor by which future spending is discounted to the present and $G_0$ is the present discounted value of government spending, which is equal, by virtue of the assumption that $\tau_t = g_t$, to the present discounted value of lump-sum taxes.

Government spending enters the intertemporal budget constraint, affecting the decisions of the family, the real equilibrium of the economy, and thus the time paths of $w_t$ and $r_t$ (and hence $R_t$). Suppose that the government demands a constant amount of resources, $g$, per capita, where $g$ is small. Using the equivalence between the decentralized and the centrally planned economy, we draw figure 2.3 to show the dynamics. The diagram is the same as figure 2.2, except that the output available for the private sector is reduced by the uniform amount $g$, accounting for the vertical shift downward of the $dk/dt = 0$ locus to $(dk/dt = 0)'$.

There is no equilibrium at low levels of the capital stock. But once there is sufficient capital to produce goods for the government, beyond $k'$, the analysis is similar to that in figure 2.2. The economy will proceed to a steady state at $E'$ with the modified golden rule capital stock, and with consumption $c^{*\prime}$ smaller by an amount $g$ than it was in the steady state in figure 2.2. In steady state government spending completely crowds out private consumption but has no effect on the capital stock.

Does a change in government spending have dynamic effects on capital accumulation? If the economy is in steady state initially, the change in government spending is reflected instantaneously in consumption with no dynamic effect on capital accumulation. If the economy is not initially in

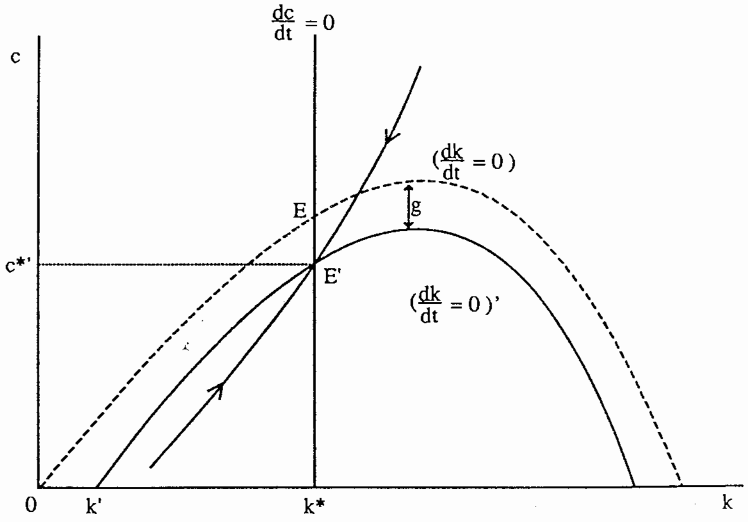

**Figure 2.3**
The effects of an increase in public spending

steady state, whether or not the change in spending has a transitory effect depends on the characteristics of the felicity function. If, for example, felicity belongs to the CARA class, there is no dynamic effect on capital accumulation.

### Debt Financing

Instead of financing itself through taxes with $\tau_t = g_t$, the government may borrow from the private sector. Government debt must pay the same rate as capital, if agents are to hold it in their portfolios. Let $b_t$ be per capita government debt. The government faces the following dynamic budget constraint:

$$\frac{db_t}{dt} + nb_t = g_t - \tau_t + r_t b_t.$$

The left-hand side is government borrowing per capita, which is equal to the increase in the per capita debt ($db_t/dt$) plus the amount of debt ($nb_t$) that can, as a result of the growing population, be floated without increasing the amount of debt per capita. The right-hand side is the excess of government outlays, consisting of its purchases of goods and services and interest payments, over its tax receipts. The flow constraint says only that the government has to borrow when its outlays exceed its tax receipts, or that it repays debt or lends to the private sector when tax receipts exceed outlays.

Integrating this budget constraint and imposing the NPG condition this time on the government (that debt not increase faster asymptotically than the interest rate) gives an intertemporal budget constraint for the government:

$$b_0 + \int_0^\infty g_t R_t \, dt = \int_0^\infty \tau_t R_t \, dt. \tag{24}$$

The present value of taxes must be equal to the present value of government spending plus the value of the initial government debt $b_0$, given the NPG condition. Equivalently, the government must choose a path of spending and taxes such that the present value of $g_t - \tau_t$, which is sometimes referred to as the primary deficit, equals the negative of initial debt, $b_0$; if the government has positive outstanding debt, it must anticipate running primary surpluses at some point in the future. For instance, it is consistent with (24) that the government maintain the initial value of the per capita debt, $b_0$, forever, running a primary surplus just large enough to pay the interest net of the amount of debt that can be financed by selling $b_0$ to each newborn person.

The presence of government debt also modifies the dynamic budget constraint of the family, which becomes

$$c_t + \frac{da_t}{dt} + na_t = w_t + r_t a_t - \tau_t, \tag{25}$$

with $a_t$ now equal to $k_t - b_{pt} + b_t$. Note that there is an implicit assumption in (25) that the family can borrow and lend at the same interest rate, $r_t$, as the government.

Integrating this budget constraint subject to the NPG condition gives the following intertemporal budget constraint:

$$\int_0^\infty c_t R_t \, dt = k_0 - b_{p0} + b_0 + \int_0^\infty w_t R_t \, dt - \int_0^\infty \tau_t R_t \, dt. \tag{26}$$

The present value of consumption must be equal to the sum of nonhuman wealth, which is the sum of $k_0 - b_{p0}$ and $b_0$, and of human wealth, which is the present value of wages minus taxes.

The government budget constraint shows that for a given pattern of government spending (and given $b_0$), the government has to levy taxes of a given present value: equivalently, the government need not run a balanced budget at every moment of time. For instance, starting from a balanced budget, it can reduce taxes at some point, borrow from the public, and raise future taxes to repay the interest and the debt.

What then is the effect of a change in the timing pattern of the taxes raised to finance a given pattern of government expenditures? The answer is given by replacing the intertemporal budget constraint of the government in (26). This gives

$$\int_0^\infty c_t R_t\,dt = k_0 - b_{p0} + \int_0^\infty w_t R_t\,dt - \int_0^\infty g_t R_t\,dt. \tag{27}$$

Equation (27) is exactly the same as equation (23). Neither taxes nor government debt appear in the budget constraint of the family. Only government spending matters. This has a strong implication: *for a given path of government spending, the method of finance, through lump-sum taxation or deficit finance, has no effect on the allocation of resources.*

The intuition for this result is obtained by looking at the intertemporal budget constraints of the government and families. A decrease in taxes, and thus a larger deficit today, must according to the government budget constraint lead to an increase in taxes later. According to the family budget constraint, the current decrease and the anticipated future increase exactly offset each other in present value, leaving the budget constraint unaffected. Families thus do not modify their paths of consumption. They willingly save the increase in current income, exactly offsetting the dissaving of the government.

This conclusion is remarkable, for it provides one instance in which, so long as the government ultimately meets its NPG condition, the size of the national debt is of no consequence, and neither is deficit finance. We will return several times to the issue of the effects of the national debt and deficit finance and study the robustness of this strong neutrality result.

### Distortionary Taxation of Capital

Distortionary taxation certainly affects the allocation of resources. Suppose that the government taxes the return to capital at the rate $\tau_k$, and remits the proceeds in lump-sum fashion to the private sector. If $r_t$ is the pre-tax rate of return on capital, $(1 - \tau_k)r_t$ is the aftertax return on capital and must also be the rate of return on private debt as capital and debt are perfect substitutes in the family's portfolio. The family's flow budget constraint is now

$$c_t + \frac{da_t}{dt} + na_t = w_t + (1 - \tau_k)r_t a_t + z_t, \tag{28}$$

where $z_t$ is the per capita lump-sum transfers (equal to the government's receipts from the taxation of capital) made to the family.

Setting up the Hamiltonian for this problem yields a modification of (19):

$$\frac{du'(c_t)/dt}{u'(c_t)} = \theta + n - (1 - \tau_k)r_t. \tag{19'}$$

Note first that the taxation of capital affects the steady state capital stock. With $r_t = f'(k_t)$, the steady state capital stock (when $dc/dt = 0$) is given by

$$k^* = f'^{-1}\left(\frac{\theta + n}{1 - \tau_k}\right).$$

The aftertax rate of return to capital will still be equal to the rate of time preference adjusted for population growth $\theta + n$; for that reason the pre-tax rate of return is higher than $\theta + n$. The marginal product of capital in steady state is accordingly higher, meaning that the steady state capital stock is lower than when capital is not taxed.

Figure 2.4 shows how the taxation of capital affects the economy. The steady state moves from $E$ to $E'$, the steady state capital stock falls from $k^*$ to $k^{*\prime}$, and the steady state rate of consumption is lower than it was in the

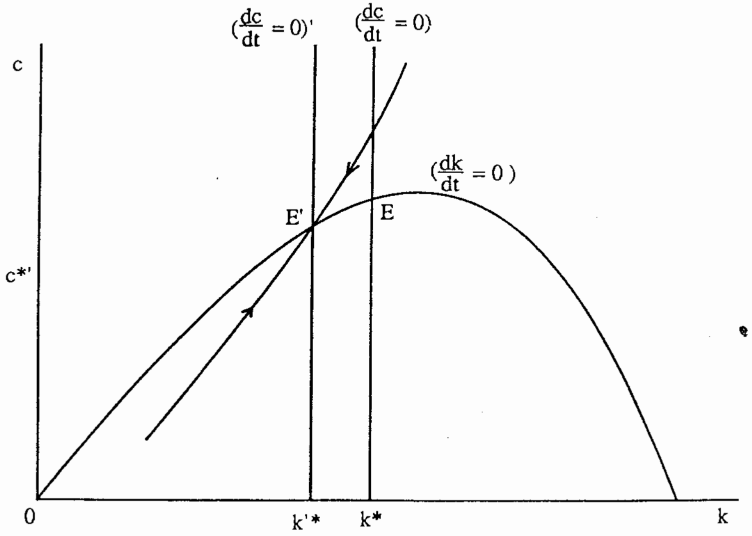

**Figure 2.4**
The effects of capital taxation

absence of distortionary taxation. Equivalently, if the government instead subsidized capital using lump-sum taxation, it could increase the steady state capital stock and level of consumption, as long as the steady state capital stock was below the golden rule level.

## 2.4 Application: Investment and Saving in the Open Economy

In this section we extend the closed economy optimizing model to the open economy. The extension not only sheds light on the optimal and actual responses of an economy to shocks, such as a reduction in productivity, but it also provides further insights into investment and saving behavior.

We extend the original model in two directions. In the closed economy model used so far, there was no cost to installing capital; whatever was saved could be added to the capital stock at no cost, and investment was purely passive. We now introduce costs of installation. This will be seen to imply that there is now both a well-defined saving decision and a well-defined investment decision.

If we maintained the assumption that the economy was closed, interest rates would have to adjust so that saving would be equal to investment at all points in time, or equivalently so that the demand for goods, consumption plus investment, would be equal to the supply of goods.[28] Instead, we open up the economy, allowing international trade in both goods and assets. It is then possible for saving and investment not to be equal at any moment of time: temporary imbalances, current account deficits, can be financed by foreign borrowing. In this way we show most clearly the separate dynamics of investment and saving.[29]

As before, there is equivalence between the command optimum and the decentralized equilibrium. We describe the command optimum in the text, and demonstrate its equivalence to the decentralized equilibrium in appendix C.

### The Command Optimum

The optimization problem is

$$\max U_0 = \int_0^\infty u(c_t)\exp(-\theta t)\,dt \tag{29}$$

subject to

$$\frac{db_t}{dt} = c_t + i_t\left[1 + T\left(\frac{i_t}{k_t}\right)\right] + \theta b_t - f(k_t), \tag{30}$$

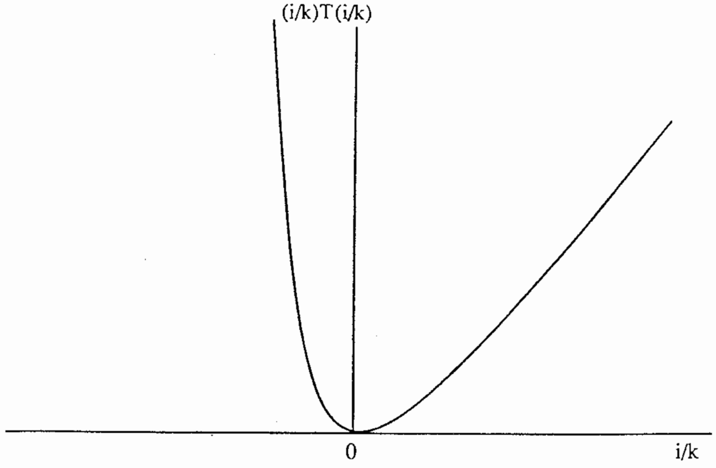

**Figure 2.5**
Costs of installation for investment

$$\frac{dk_t}{dt} = i_t, \tag{31}$$

$$T(0) = 0,$$

$$T'(\cdot) > 0,$$

$$2T'(\cdot) + \frac{i}{kT''(\cdot)} > 0.$$

All variables are in per capita terms, and population is assumed here to be constant; $b$ denotes per capita debt. The felicity and production functions $u(\cdot)$ and $f(\cdot)$ have the same properties as in earlier sections.

There are two changes from the previous analysis. First, there are now costs of installing investment goods.[30] It takes $i[1 + T(\cdot)]$ units of output to increase the capital stock by $i$ units. The amount $T(\cdot)$ per unit of investment is used up in transforming goods into capital. The properties of $T(\cdot)$ make the installation cost function $(i/k)T(\cdot)$, shown in figure 2.5, nonnegative and convex, with a minimum value of zero when investment is equal to zero: both investment and disinvestment are costly. For simplicity, we assume that there is no depreciation.[31]

The other difference is that the economy can now borrow and lend freely abroad at the constant world interest rate $\theta$.[32] This implies the flow budget constraint (30): the change in foreign debt ($db/dt$) is equal to spending (on consumption, investment, and interest payments) minus output. The change in foreign debt is the current account deficit, so that (30) is equivalent to the statement that the current account deficit is equal to the excess of absorption over production.

There is a simple relation among the current account deficit, saving, and investment that will be useful later. A brief refresher in national income accounting identities and definitions may be useful at this point. The current account deficit is equal to the change in foreign debt, which is in turn equal to interest payments minus net exports of goods, the trade surplus ($nx$):

$$\frac{db}{dt} = \theta b - nx.$$

Per capita GDP and GNP are given by

$$\text{GDP} = c + i[1 + T(\cdot)] + nx,$$

$$\text{GNP} = \text{GDP} - \theta b.$$

Saving is equal to GNP minus consumption:

$$s = \text{GNP} - c = i[1 + T(\cdot)] + nx - \theta b = i[1 + T(\cdot)] - \frac{db}{dt}$$

so that

$$\frac{db}{dt} = \text{current account deficit} = i[1 + T(\cdot)] - s.$$

The maximization problem as now stated has a simple solution. It is again a Ponzi-like solution, but now on the part of the central planner vis-à-vis the rest of the world. The country should borrow until the marginal utility of consumption is equal to zero, and then borrow further to meet interest payments on its debt. It is unlikely that the lenders would be willing to continue lending if the country's only means of paying off its debt were to borrow more. Accordingly, we impose the NPG condition[33]

$$\lim_{t\to\infty} b_t \exp(-\theta t) = 0. \tag{32}$$

To solve the intertemporal problem, we set up the present value Hamiltonian:

$$H_t = \left[u(c_t) - \mu_t\left\{c_t + i_t\left[1 + T\left(\frac{i_t}{k_t}\right)\right] + \theta b_t - f(k_t)\right\} + \mu_t q_t i_t\right]\exp(-\theta t). \tag{33}$$

The costate variables on the flow budget constraint (30) and the capital accumulation equation (31) are $-\mu_t \exp(-\theta t)$ and $\mu_t q_t \exp(-\theta t)$.[34]

Necessary and sufficient conditions for a maximum are

$$u'(c_t) = \mu_t \qquad (\text{from } H_c = 0), \tag{34}$$

$$1 + T\left(\frac{i_t}{k_t}\right) + \left(\frac{i_t}{k_t}\right)T'\left(\frac{i_t}{k_t}\right) = q_t \qquad (\text{from } H_i = 0), \tag{35}$$

$$\frac{d[-\mu_t \exp(-\theta t)]}{dt} = \theta\mu_t \exp(-\theta t), \tag{36}$$

$$\frac{d[\mu_t q_t \exp(-\theta t)]}{dt} = -\left\{\mu_t\left[f'(k_t) + \left(\frac{i_t}{k_t}\right)^2 T'\left(\frac{i_t}{k_t}\right)\right]\right\}\exp(-\theta t), \tag{37}$$

$$\lim_{t\to\infty} -\mu_t b_t \exp(-\theta t) = 0, \tag{38}$$

$$\lim_{t\to\infty} \mu_t q_t k_t \exp(-\theta t) = 0. \tag{39}$$

Equations (36) and (37) are the Euler equations associated with $b$ and $k$, respectively. Equations (38) and (39) are the transversality conditions associated with $b$ and $k$, respectively. We are now ready to characterize the solution. We start with consumption.

### Consumption

Carrying out the differentiation in (36), we obtain

$$\frac{d\mu_t}{dt} = 0, \tag{40}$$

which implies that $\mu$ is constant. In turn, this implies from (34) that consumption is constant on the optimal path. That is precisely what should be expected given the findings in sections 2.1 and 2.2 on the effects of the relationship between the interest rate and the rate of time preference on the profile of the consumption path.

To obtain the level of consumption, we integrate the flow constraint (30) using condition (38), which yields

$$\int_0^\infty c_t \exp(-\theta t)\,dt = \int_0^\infty \left\{f(k_t) - i_t\left[1 + T\left(\frac{i_t}{k_t}\right)\right]\right\}\exp(-\theta t)\,dt - b_0$$
$$= v_0. \tag{41}$$

The present discounted value of consumption is equal to net wealth at time 0, $v_0$, the present discounted value of net output [the contents of the braces in (41)] minus the initial level of debt. Since consumption is constant, (41) implies that[35]

$$c_t = c_0 = \theta v_0. \tag{42}$$

### Investment

Equation (35) contains a very strong result, namely, that the rate of investment (relative to the capital stock) is a function only of $q_t$, which is the shadow price in terms of consumption goods of a unit of installed capital. Equation (35) implies a relation $q = \Psi(i/k)$, with $\Psi' > 0$ and $\Psi(0) = 1$. Thus we can define an inverse function $\varphi(\cdot)$ such that $i/k = \varphi(q)$. From the properties of $\Psi(\cdot)$, it follows that $\varphi' > 0$ and $\varphi(1) = 0$. Replacing in (31) gives

$$\frac{dk_t}{dt} = i_t = k_t \varphi(q_t), \qquad \varphi'(q) > 0,\ \varphi(1) = 0. \tag{43}$$

Investment is, from (43), an increasing function of $q$, the shadow price of capital. At the margin the planner equates the value of an addition to the capital stock with its marginal cost, which rises with the rate of investment. It makes sense to incur the higher marginal cost of investing faster only when the shadow value of capital is higher. Note that the rate of investment is zero when $q = 1$, when the shadow price of capital is the same as that of goods "on the hoof," so that positive rates of investment require $q > 1$. Note finally that the level of $q$ determines the rate of investment relative to the capital stock, $i_t/k_t$.

What in turn determines $q$? From (37), given (40),

$$\frac{dq_t}{dt} = \theta q_t - f'(k_t) - \varphi(q_t)^2 T'[\varphi(q_t)]. \tag{37'}$$

Integrating (37′) subject to (39),[36]

$$q_t = \int_t^\infty \{f'(k_v) + \varphi(q_v)^2 T'[\varphi(q_v)]\} \exp[-\theta(v - t)]\, dv. \tag{44}$$

The shadow price of capital is equal to the present discounted value of future marginal products. Marginal product is itself the sum of two terms: the first is the marginal product of capital in production; the second is the reduction in the marginal cost of installing a given flow of investment due to the increase in the capital stock (because the installation cost depends on the ratio of investment to capital). The higher the current or future expected marginal products or the lower the discount rate, the higher are $q$ and the rate of investment.[37]

The most significant feature of (44) is that $q$, and thus the rate of investment, does not depend at all on the characteristics of the utility function or on the level of debt. The investment decision is independent of the saving or consumption decisions in this open economy framework with an exogenous real interest rate.

### Saving, Investment and the Current Account

Saving is given by

$$s_t = f(k_t) - c_t - \theta b_t.$$

From the derivation of consumption above, $c_t = \theta v_t$ so that

$$s_t = f(k_t) - \theta \int_t^\infty \left\{ f(k_z) - i_z\left[1 + T\left(\frac{i_z}{k_z}\right)\right]\right\} \exp[-\theta(z - t)]\, dz. \tag{45}$$

Thus saving is high when output is high compared to future expected output. The other distinctive result is that saving is independent of the level of debt: the equality of the marginal propensity to consume and of the interest rate implies that a higher level of debt leads to equal decreases in income and consumption, leaving saving unaffected.

Since the current account surplus is equal to saving minus investment, neither of which is affected by the stock of debt, the current account is also independent of the stock of debt.

### Steady State and Dynamics

The dynamic system characterizing the behavior of the economy is recursive, with (43) and (37′) determining investment, capital, and output. The level of consumption and debt dynamics are then determined by (42) and (30).

### Investment and Capital

In steady state $dk/dt = dq/dt = 0$. Accordingly, from (31), from $\varphi(1) = 0$, and from (37′),

$$q^* = 1, \qquad f'(k^*) = \theta, \tag{46}$$

where the asterisks denote steady state values.

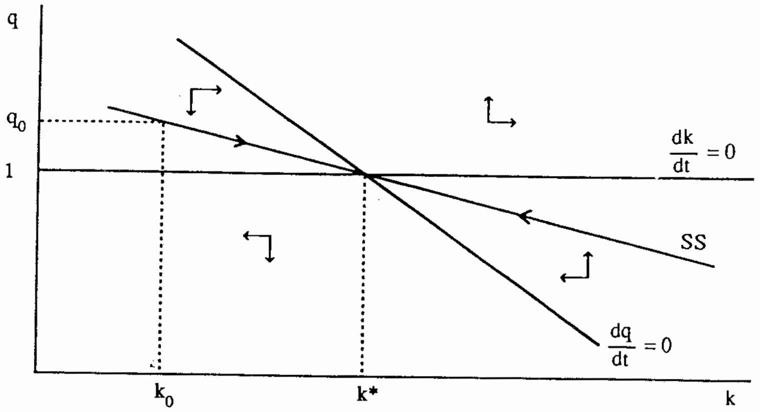

**Figure 2.6**
Dynamics of investment and capital

In steady state the rate of investment is zero.[38] The shadow price of capital must therefore be equal to its replacement cost, or $q = 1$; in turn, the marginal product of capital has to be equal to the interest rate, which is itself equal to the rate of time preference.

We limit our analysis of the dynamics of investment and capital to a neighborhood around the steady state. To do so, we linearize (43) and (37′) around $q^* = 1$ and $k^*$:

$$\begin{bmatrix} \frac{dk}{dt} \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} 0 & k^*\varphi'(1) \\ -f''(k^*) & \theta \end{bmatrix} \begin{bmatrix} k - k^* \\ q - 1 \end{bmatrix}. \tag{47}$$

Figure 2.6 gives the phase diagram corresponding to (47). The $dk/dt = 0$ locus is horizontal at $q = 1$; the $dq/dt = 0$ locus is downward sloping.[39] The arrows indicate directions of motion. There is therefore a unique path converging to the steady state, the downward-sloping path $SS$.[40]

The dynamics of investment are implied by the saddle point path $SS$. Given an initial capital stock $k_0$, the initial value of $q$, $q_0$ is read off $SS$, and the associated level of investment follows from (43). Since $q_0$ in this case exceeds unity, capital accumulates over time. Output increases and so does net output, which is equal to $f(k) - i[1 + T(i/k)]$: output increases while investment decreases over time.

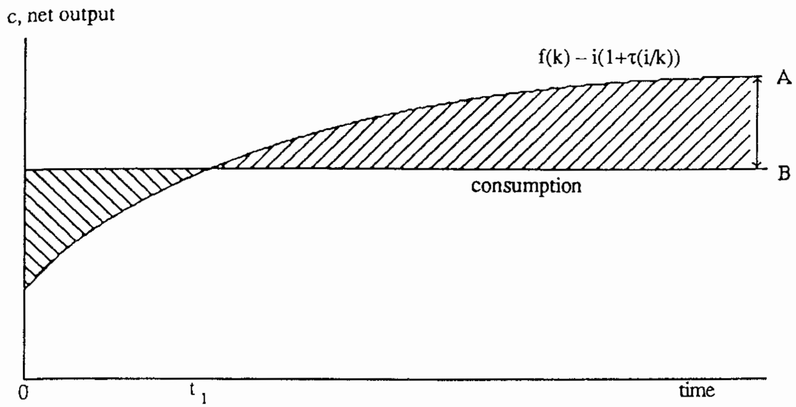

**Figure 2.7**
Consumption, net output, trade and current accounts

## Consumption and Debt

We have already seen that the level of consumption is constant, determined by the path of net output (which is itself determined by the path of investment) and by the initial stock of debt.

Figure 2.7 shows a path of net output that increases over time as the capital stock increases to its steady state level. We will now determine the level of consumption in figure 2.7. Assume that the initial stock of debt $b_0$ was zero. The constant level of consumption must then, from (41), be such that the present discounted value of net output minus consumption is zero, or equivalently that the present discounted value of current and future trade surpluses is zero.[41] Graphically, the discounted values of the two hatched areas in figure 2.7 must be equal and opposite in sign; the level of consumption is determined by drawing a horizontal line such that the two areas are equal in present value.

In figure 2.7 net output increases over time. Net output accordingly starts out below and eventually exceeds consumption. The initial excess of consumption over net output is achieved by foreign borrowing, or by running a current account deficit. Debt accumulates during this phase. Eventually, net output rises sufficiently so that the trade balance shows a surplus. In steady state the current account must be balanced. The trade surplus is offset by interest payments on debt. The steady state level of debt $b^*$ is positive and such that $\theta b^*$ is equal to $AB$, the trade surplus, in figure 2.7. The presence