of the debt reflects the decision to consume at a rate above the level of net output early in time.

Productivity Shocks and the Current Account

The paths of consumption and the current account in the preceding analysis can serve as a baseline for the analysis of the effects of shocks to productivity on the economy. Suppose that output is given by

$$y = (1 - z_0)f(k) - z_1.$$

Here z_0 is a multiplicative shock, and z_1 is an additive shock. Reflecting the experience of oil shocks in the 1970s, we take the shocks to be adverse, and consider increases in either z_0 or z_1 that reduce output given the capital stock.[42]

A Permanent Additive Shock

We start by considering an unexpected permanent increase in z_1, from a value of zero, with the economy initially in steady state.[43] An increase in z_1 has no effect on the marginal product of capital and thus no effect on investment and the capital stock. Since the change is both unexpected and permanent, the increase in z_1 leads to an unexpected and permanent reduction in net output by the same amount, z_1. From (41) it follows that consumption falls by exactly the same amount. Saving therefore remains unchanged. Further, with both savings and investment unchanged, the current account is unaffected by the productivity shock.

In this case of an unexpected permanent reduction in z_1, the economy takes its losses immediately, and with no further consequences for the allocation of resources.[44]

A Transitory Additive Shock

Suppose now that z_1 increases unexpectedly but temporarily at time 0 for a period of length T. There is still no effect on investment, but there will now be a change in saving and the current account. The change in the present discounted value of net output is given by

$$-z_1 \int_0^T \exp(-\theta t)\, dt,$$

or equivalently

$$-z_1\theta^{-1}[1 - \exp(-\theta T)]$$

so that the change in consumption is given by $-z_1[1 - \exp(-\theta T)]$.

This change in consumption is permanent. If T is small, the change in consumption is also small. Agents cut consumption only a little, and most of the decrease in output translates into a reduction in saving and a current account deficit. After output returns to normal, the economy runs a permanent trade surplus to pay for the increased interest payments on the debt. If T is large, the change in consumption is larger, the reduction in saving and the increase in debt smaller. As T tends to infinity, we obtain the same results as in the permanent case.

A Permanent Multiplicative Shock

Finally, let z_0 increase from zero to a positive value at $t = 0$. Because the marginal product of capital is $(1 - z_0)f'(k)$, a change in z_0 affects investment. We start by analyzing the effects of the change on investment and output.

As figure 2.8 shows, the increase in z_0 shifts the $dq/dt = 0$ locus to the left. The $dk/dt = 0$ locus is unaffected. The steady state of the economy shifts from E to E'. At E' the steady state capital stock $k^{*\prime}$ is lower than k^*, the initial steady state capital stock; the marginal product of capital $(1 - z_0)f'(k)$ is again equal to θ.

The new saddle point path is SS'. With the initial capital stock given by k^*, the path of adjustment is composed of a jump at time 0 from E to A, and a movement over time from A to E'. The rate of investment is negative on

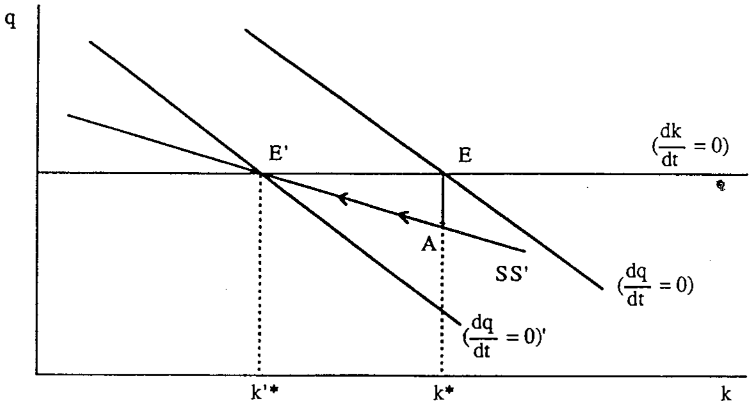

Figure 2.8
Effects of an adverse supply shock on investment and capital

the adjustment path, returning to zero as the economy moves to the new lower steady state capital stock.

Net output, which is equal to $(1 - z_0)f(k) - i[1 + T(i/k)]$, may either increase or decrease initially, depending on whether the fall in output when z_0 increases is larger or smaller than the decline in investment. In the long run, however, the effect is unambiguous. As investment returns to zero and the capital stock falls to a lower level, net output must be lower in the new steady state, because the initial effect of the adverse shock is compounded by lower capital accumulation.

In figure 2.9 we show net output falling initially, and then declining further to its new steady state level. The new level of consumption is determined again by the condition that the present value of the hatched areas above and below it be equal.

With net output above consumption immediately after the shock, the economy is saving in anticipation of lower net output later. In figure 2.9 the economy runs current account and trade surpluses immediately after the shock and becomes a net owner of foreign assets in steady state.

These examples show that there is no simple relation between adverse supply shocks and the current account even in the simple model developed in this section. What happens depends on the nature of the shocks affecting the economy, for example, whether they are additive or multiplicative, temporary or permanent. Sachs (1981) has used a closely related model to study the effects of the oil shocks of the 1970s on the current accounts of different groups of countries. He argues that the response of the industrializing developing countries, which borrowed extensively abroad during that period, conforms roughly to the predictions of the model.

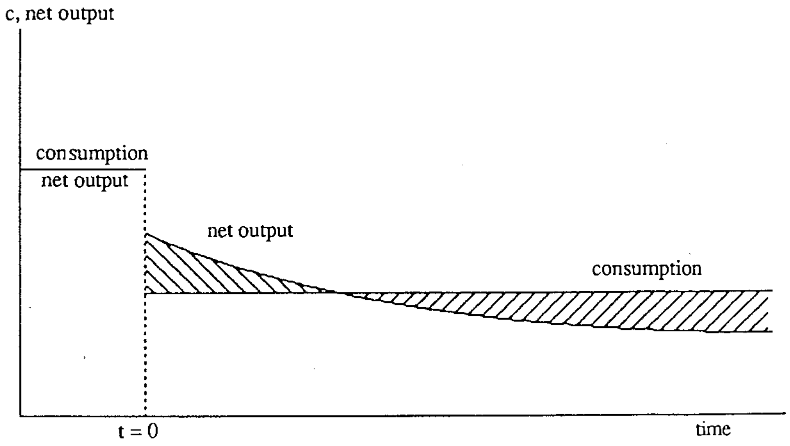

Figure 2.9
Dynamic effects of an adverse multiplicative shock

2.5 The Utility Function

The assumption that the utility function is additively separable, with exponential discounting, produces strong results. In particular, because the modified golden rule relation (11) fixes the steady state real interest rate, the model of this chapter implies that no policy changes or shocks to the production function can affect the steady state aftertax real interest rate.

There is, however, no strong reason, beyond analytical convenience, to assume additive separability or a constant rate of time preference. Marginal utility of current consumption may well depend on past consumption through habit or through boredom effects. One may well have a rate of time preference that changes through life or, say, between summers and winters. What happens when we allow for such complications? This is the question we briefly explore in this last section.

Relaxation of the assumption of additive separability may lead to far more complex dynamics. Assume, for instance, the following form of the felicity function at time t:

$$u(c_t, z_t),$$

where z_t depends on past rates of consumption. The assumption here is that the history of consumption affects the marginal utility of current consumption. Ryder and Heal (1973) have shown that, with only this modification to the standard Ramsey model, optimal paths may overshoot the modified golden rule steady state, and that oscillatory approaches to the steady state are possible.

In this section we continue, however, to assume that the felicity function takes the form $u(c_t)$ and examine, instead, the assumption of a constant rate of time preference.[45] We first show a further implication of constant rates of time preference and then explore the rationale for that formulation and present an alternative representation.

Differences in Rates of Time Preference

We have assumed until now that all families have the same discount rate θ. There is no reason why this should generally be so. Consider the alternative in which there are m different types of families, ordered by decreasing impatience, with rates of time preference $\theta_1 > \cdots > \theta_m > 0$. The economy is in other respects identical with that of section 2.2, except that for expositional simplicity we assume zero population growth.

We now show that in steady state the interest rate r must be equal to the lowest rate of time preference θ_m. Suppose that this is not the case, and that r is smaller than θ_m. Then with r smaller than all rates of time preference, all families have decreasing consumption over time, by equation (19); but if all families have decreasing consumption and there is no population growth, the economy cannot be in steady state with constant aggregate consumption.

Suppose, instead, that r is greater than θ_m. All families with discount rate smaller than r have increasing consumption, and others have decreasing consumption. The share of total consumption accounted for by families with increasing consumption must be increasing over time. Indeed, the share of total consumption accounted for by the families with the lowest discount rate must eventually tend to one. Total consumption must therefore eventually be increasing; this is again inconsistent with being in steady state.

In steady state therefore the interest rate r equals θ_m.[46] The consumption of the most patient consumers is constant. The consumption of all other families is declining so that eventually their share in total consumption is zero. Slow and steady wins the race, and all the wealth; not only do the most patient families own all physical capital, but they also "own" the human capital of others who pay over all their labor income in return for past borrowing.[47] The model paints a somber, though unrealistic, picture of the dynamics of income and wealth distribution.

A slightly less extreme result is obtained if consumers are prohibited from borrowing against labor income and thus are constrained to have nonnegative financial wealth. The steady state will still have $r = \theta_m$. The most patient families will hold all nonhuman wealth; all others will have a level of consumption equal to their labor income.[48] This result, though less drastic, is still not a good description of income distribution dynamics.[49]

There are many simplifications in this model, including the absence of uncertainty and the existence of infinitely lived families. These are possible directions in which to search for better models of income distribution dynamics. Another direction, to which we turn below, is the specification of preferences. What happens when we relax the assumption that the discount rate is constant?

Calendar Time, Time Distance, and Time Consistency

Suppose that, instead of the assumption of a constant rate of time preference the utility integral is given by

$$U_s = \int_s^{\infty} u(c_t)D[t, t-s, x(t)]\, dt. \tag{48}$$

Utility is a weighted integral of felicities at different times, with the weighting function, $D(\cdot)$, referred to as a discount function. In (48) we make the discount function potentially a function of calendar time, t, of *time distance*, $t - s$, and possibly other variables, $x(t)$, for instance, the rate of consumption itself. Note that the formulation (1) makes the discount factor between any two periods purely a function of time distance: the rate of discount applied to utility for any particular number of years (say, T) in the future is always the same [in this case $\exp(-\theta T)$].

In any optimal program the marginal rate of substitution between consumption at any two dates is equal to the marginal rate of transformation. Using (48),[50] we now characterize the optimal program. Consider two planning dates, τ_1 and τ_2, and two points in time about which plans are made, t_1 and t_2: assume that

$$t_2 > t_1 > \tau_2 > \tau_1.$$

As of planning date τ_1, the marginal rate of substitution between consumption at time t_1 and consumption at time t_2 is given by

$$\frac{u'(c(t_1))D(t_1, t_1 - \tau_1)}{u'(c(t_2))D(t_2, t_2 - \tau_1)}. \tag{49}$$

Now consider the same marginal rate of substitution between consumption at time t_1 and consumption at time t_2 viewed as of planning date τ_2, $\tau_2 > \tau_1$:

$$\frac{u'(c(t_1))D(t_1, t_1 - \tau_2)}{u'(c(t_2))D(t_2, t_2 - \tau_2)}. \tag{50}$$

Comparison of (49) and (50) indicates that since $D(t, t - \tau_1)$ is generally different from $D(t, t - \tau_2)$, there is no reason for the rates of substitution between consumptions at times t_1 and t_2 to be the same from the two different planning dates. This implies that the optimal plan chosen at time τ_1 will no longer be optimal as of time τ_2. The optimal plan is therefore *time inconsistent*: the family's optimal plan changes over time even though no new nformation becomes available.

There are now two issues: First, under what restrictions on the discount unction $D(\cdot)$ is the optimal plan time consistent? And second, what happens vhen the optimal plan is time inconsistent?

For the optimal plan to be the same as of time τ_1 and time τ_2, the marginal ates of substitution (49) and (50) must be the same. This can happen if the discount function $D(\cdot)$ is *either* an exponential function of time distance, $\exp[-\theta(t-s)]$ [check that this form ensures that the rates of substitution (49) and (50) are the same], *or* purely a function of calendar time or calendar values of other variables.[51] The first instance gives one possible rationale for assuming exponential discounting.

Dealing with Inconsistency

What happens, though, if people do have a discount function that leads to time inconsistency?[52] This question can be handled at many levels, but we do not dig deeply.[53] One possibility is that of *precommitment*. Consumers, having solved their optimal plan at time $t = 0$, may find a way of committing themselves to the plan to prevent what they then (at $t = 0$) would regard as backsliding. For instance, they could commit themselves to a savings path by entering a savings plan.

Another possibility is that consumers recognize that their tastes will be changing and make their plans assuming that they will at each future moment follow their tastes of that moment. They then choose a consistent plan in the sense that all future actions are correctly taken into account in the planning process.[54] If their discount function is only a function of time distance, this will lead them to act as if they had a constant rate of time preference through life.

This gives two possible rationales for assuming exponential discounting. The first is that exponential discounting leads to optimal programs that are time consistent. The other is that even if families do not exponentially discount the future but behave in time-consistent fashion, they may act as if they had exponential discounting. As we have seen, however, exponential discounting is not the only form of discounting that leads to time consistency. We now examine a formulation of the discount function in which the discount rate depends on the level of consumption. This formulation leads to a rate of time preference that changes through time but still implies time consistency of the optimal program.

Dependence of the Discount Rate on Utility

Uzawa (1968) considered the possibility that the rate of time preference depends on the level of utility or consumption. Uzawa's utility functional is

$$\int_0^\infty u(c_t) \exp\left\{-\int_0^t \theta[u(c_v)]\, dv\right\} dt.$$

The innovation is that the instantaneous rate of time preference is a function of the current level of utility, and thus of consumption.[55] Uzawa specified that

$$\theta'(\cdot) > 0. \tag{51}$$

The implication of (51) is that a higher level of consumption at time v increases the discount factor applied to utility at and after v. In steady state a higher level of consumption implies a higher rate of time preference. The assumption $\theta'(\cdot) > 0$ is difficult to defend *a priori*; indeed, we usually think it is the rich who are more likely to be patient. Assumption (51) is, however, needed for stability: if the rate of time preference fell with the level of consumption, the rich would become richer over time. That problem does not arise when, as in (51), the rate of time preference increases with the level of consumption.

We do not analyze the dynamics of this growth model but briefly characterize the steady state. Setting up the full model as in section 2.1, and deriving optimality conditions for the family, we obtain[56]

$$[u'(c_t) - \lambda_t] - \frac{\theta'[u(c_t)]u'(c_t)}{\theta[u(c_t)]}\{u(c_t) - \lambda_t[f(k_t) - nk_t - c_t]\} = 0,$$

$$\frac{d\lambda_t/dt}{\lambda_t} = \theta[u(c_t)] + n - f'(k_t), \tag{52}$$

$$\lim k_t\lambda_t \exp\left\{-\int_0^t \theta[u(c_v)]\, dv\right\} = 0. \tag{53}$$

The first equation gives the relation between the costate variable and consumption. Note that the dependence of the discount rate considerably complicates this relation, which we shall not discuss further. Note also that if $\Theta(\cdot)$ is constant, this relation reduces to $u'(c) = \lambda$, as in the Ramsey model. Equation (52) gives the relation between the rate of change of the costate variable the discount rate, and the marginal product; this relation is as in the Ramsey model. Equation (53) is the standard transversality condition.

The other equation is the capital accumulation equation:

$$\frac{dk_t}{dt} = f(k_t) - nk_t - c_t.$$

In steady state $d\lambda/dt = dk/dt = 0$ so that

$$\theta[u(c^*)] = f'(k^*) - n,$$

$$c^* = f(k^*) - nk^*.$$

These two loci are drawn in figure 2.10.

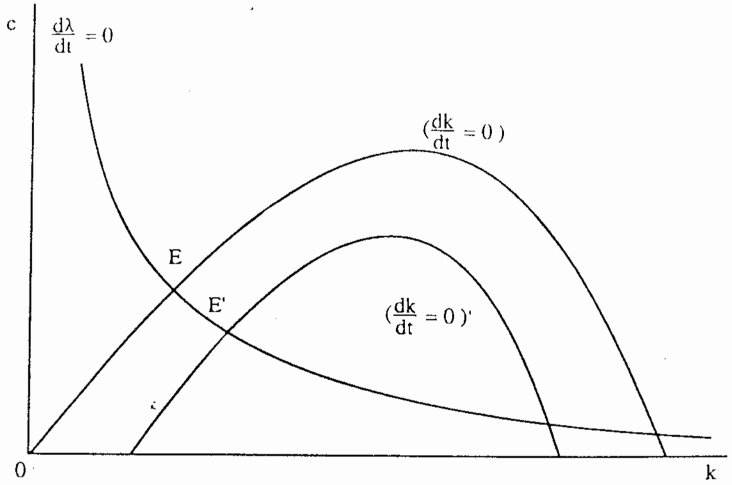

Figure 2.10
Dynamics with endogenous time preference

The $dk/dt = 0$ locus is the same as in section 2.1. The $d\lambda/dt = 0$ locus however, is now downward sloping rather than vertical. The saddle point steady state equilibrium is at point *E*.

Consider now an additive productivity shock, an increase in z_1, that shifts the $dk/dt = 0$ locus down uniformly. The new equilibrium is at *E'*. The steady state capital stock rises so that the reduction in consumption caused by lower productivity is compensated for by an increase in capital. The rate of time preference and the real interest rate are lower in the new equilibrium at *E'* than they were at *E*.

The results in figure 2.10 contrast sharply with those that would occur in the model of section 2.1. In that case the fall in productivity would leave the steady state real interest rate unaffected and would result in a reduction in consumption exactly equal to the decrease in output.

Returning to the issue that motivated our look at the Uzawa formulation, consider the situation where families have different discount rate functions. In steady state all discount rates will be equal. This implies a distribution of consumption across families and an associated steady state distribution of wealth. Families with more patience, in the sense that at a given level of consumption their rate of time preference is lower, achieve higher steady state wealth and consumption.

This specification avoids the pathological results of the constant discount rate case. Nonetheless, the Uzawa function, with its assumption $\theta'(\cdot) > 0$, is not particularly attractive as a description of preferences and is not recommended for general use. A nondegenerate steady state, when individual tastes differ, can also be achieved by assuming that agents have finite lives; this is a more plausible avenue which we develop in chapter 3.

Appendix A: Ruling Out Explosive Paths in the Ramsey Model

To show that the saddle point path *DD* in figure 2.2 is the optimal path, suppose that the initial capital stock is k_0, $0 < k_0 < k^*$. Consider any trajectory that starts above point *D*, at *D'*, say. This path implies that the economy reaches zero capital in finite time. The proof turns on the fact that on such a path d^2k/dt^2 eventually becomes negative. Differentiating (2) gives

$$\frac{d^2k}{dt^2} = [f'(k) - n]\left(\frac{dk}{dt}\right) - \frac{dc}{dt} < 0, \qquad \text{as} \quad \frac{dc}{dt} > 0, \quad f'(k) - n > 0.$$

Thus $k_t = k_0 + \int_0^t (dk_v/dv)\, dv$ will reach zero in finite time.

Note that *c* is rising on the path starting at *D'* all the time until it hits the axis at point *B*. But when the path reaches *B*, *k* is zero, and the economy has to move to the origin. Thus *c* has to jump from a positive value to zero. But such a jump violates the necessary condition (7'), and it thus cannot have been optimal to start at *D'*.

Consider, alternatively, a trajectory starting below *D*, for example, at *D''*. This path converges asymptotically to *A*. But such a path violates the transversality condition. At points close to *A*, *k* is approximately constant, whereas from (7') and $k > k_g$,

$$\frac{du'(c)/dt}{u'(c)} = \theta + n - f'(k) > \theta.$$

Thus as *t* tends to infinity and the trajectory approaches *A*, the transversality condition is violated.

Similar arguments apply if the initial capital stock is larger than k^*. It follows that the saddle point path *DD* is the unique path that satisfies conditions (2), (7'), and (8).

Appendix B: Local Behavior of Capital around the Steady State in the Ramsey Model

The characteristic equation associated with equation (15) is

$$x^2 - \theta x - \beta = 0.$$

It has two roots:

$$\lambda \equiv \frac{\theta - \sqrt{\theta^2 + 4\beta}}{2} < 0$$

and

$$\mu \equiv \frac{\theta + \sqrt{\theta^2 + 4\beta}}{2} > 0.$$

Thus paths that satisfy equation (15) are given by

$$k_t - k^* = c_0 \exp(\lambda t) + c_1 \exp(\mu t),$$

where c_0 and c_1 are arbitrary constants.

As k_0 is given from history, c_0 and c_1 must satisfy

$$k_0 - k^* = c_0 \exp(0) + c_1 \exp(0) = c_0 + c_1.$$

In addition, as μ is positive, c_1 must be equal to zero for k to converge to k^*. Thus $c_1 = 0$ and $c_0 = k_0 - k^*$. This implies in turn that

$$k_t = k^* + (k_0 - k^*) \exp(\lambda t).$$

Appendix C: Command Optimum and Decentralized Equilibrium in the Open Economy Model

We show here the equivalence of the command optimum and the decentralized competitive equilibrium in the open economy model of section 2.4. For notational simplicity, we assume that there are as many firms as families so that the same symbol denotes the ratio of a variable per capita or per firm.

The structure of the economy is the following: Firms rent labor services in the labor market but own the capital stock; they finance investment through retained earnings. Families supply labor services and own the firms, receiving profits net of investment expenses. They allocate their income between consumption and saving, where saving takes the form of lending to the rest of the world.[57]

Value Maximization by Firms

For simplicity, we do not explicitly model the labor market. Labor is supplied inelastically so that labor market equilibrium implies that each firm hires one worker, paying wages of $\{w_t\}$ $t = [0, \infty)$. The decision problem of a representative firm at time zero is then to choose the time path of investment that maximizes the present discounted value of cash flows:

$$\max V_0 = \int_0^{\infty} \left\{ f(k_t) - i_t\left[1 + T\left(\frac{i_t}{k_t}\right)\right] - w_t \right\} \exp(-\theta t)\, dt \tag{C1}$$

subject to $dk_t/dt = i_t$ and the same technology as the central planner.

By letting $q_t \exp(-\theta t)$ be the Lagrange multiplier associated with the capital accumulation equation and setting up a present value Hamiltonian, the first-order conditions lead to equations identical to (43), (37'), and (44). Firms invest until the marginal cost of investment is equal to the shadow value of installed capital, q. This shadow value is itself equal to the present discounted value of future marginal products. Firms choose the same path of investment and capital accumulation as the central planner.

Given our assumption that firms finance investment through retained earnings, dividends paid by firms are therefore equal to net cash flows[58]

$$\pi_t = f(k_t) - i_t\left[1 + T\left(\frac{i_t}{k_t}\right)\right] - w_t. \tag{C2}$$

Utility Maximization by Families

Each family supplies one unit of labor inelastically, receiving wage w_t and dividends π_t. Its only decision problem is to choose a path of consumption that maximizes

$$U_0 = \int_0^{\infty} u(c_t) \exp(-\theta t)\, dt. \tag{C3}$$

It can borrow and lend on the world market at the rate θ. The dynamic budget constraint is therefore

$$\frac{db_t}{dt} = c_t + \theta b_t - \pi_t - w_t. \tag{C4}$$

To this we add the NPG condition:

$$\lim_{t\to\infty} b_t \exp(-\theta t) = 0. \tag{C5}$$

The solution to this maximization problem is given by

$$c_t = c_0 = \theta \int_0^{\infty} (\pi_t + w_t) \exp(-\delta t)\, dt. \tag{C6}$$

Replacing π_t by its value from (C2) in (C6) gives the same path of consumption as equation (42). Families will choose the same path of consumption as the central planner.

Appendix D: Saddle Point Equilibrium in the Linearized (k, q) System

Equation (47) linearizes the dynamic system that describes the behavior of q and k around the steady state values. The solution to such a linear system is given by

$$\begin{aligned} k_t - k^* &= c_{11} \exp(y_1 t) + c_{12} \exp(y_2 t), \\ q_t - 1 &= c_{21} \exp(y_1 t) + c_{22} \exp(y_2 t), \end{aligned} \tag{D1}$$

where y_1 and y_2 are the roots of the characteristic equation associated with (47), namely,

$$\begin{vmatrix} \theta - y & k^*\varphi'(1) \\ -f''(k^*) & \theta - y \end{vmatrix} = 0,$$

and where $[c_{11}\ c_{21}]$ and $[c_{12}\ c_{22}]$ are eigenvectors associated with each of these two roots.

The roots are given by

$$y = \frac{\theta \pm \sqrt{\theta^2 - 4f''(k^*)k^*\varphi'(1)}}{2}.$$

Both roots are real, with one root negative and the other positive. The positive root exceeds θ.

Denote the negative root by y_1. The eigenvector associated with y_1 is given by

$$\begin{bmatrix} -y_1 & k^*\varphi'(1) \\ -f''(k^*) & \theta - y_1 \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = 0, \tag{D2}$$

so that $c_{21} = \{y_1[k^*\varphi'(1)]^{-1}\}c_{11}$. Examining (A7), we see that for the path that converges to $(k^*, 1)$, both c_{12} and c_{22} must be equal to zero. (Zero is always an eigenvector.)

To calculate the constants c_{11} and c_{21}, note that at time zero, the first row of (D1) is

$$k_0 - k^* = c_{11}. \tag{D3}$$

Replacing the c's by their values in (D1) gives the converging path for k and q.

On all paths other than the converging path, c_{12} and/or c_{22} are different from zero. Thus q and k eventually increase at rate y_2. This implies that qk eventually increases at rate no less than y_2, which is itself greater than θ. Thus they all violate the transversality condition (39).

Of course, the proof that the transversality condition is violated on all but the saddle point path in the linearized system does not establish the fact that the paths of the original system that are not saddle point paths explode at a rate greater than θ. A complete proof requires a characterization of the dynamics of the original nonlinear system along the lines of the proof presented in appendix A.

Problems

1. *The Solow growth model. (This follows Solow 1956.)*

(a) Consider an economy with a population growth rate equal to n, with constant returns to scale in production, and in which individuals save a constant fraction, s, of their income. Show that the differential equation describing the behavior of the capital stock per capita is given by

$$\frac{dk}{dt} = sf(k) - nk,$$

where $f(\cdot)$ is the production function per capita and s is the savings rate.
(b) Characterize the steady state capital stock per capita in this model.
(c) Examine the stability of the system, and characterize the adjustment of the capital stock toward its steady state.
(d) Can a constant saving rate along the path of adjustment be consistent with intertemporal utility maximization by infinitely long-lived individuals?
(e) Assume that factor markets are competitive. Show that the savings rate that leads to the golden rule capital stock is equal to the share of capital in production. Explain.

2. *Growth with exogenous technological progress.*

Suppose that, in a Ramsey economy, production is given (as in note 13) by the function

$$Y_t = F(K_t, \exp(\phi t)N_t),$$

where ϕ is the constant and exogenous rate of technical progress. Assume that the population grows at rate n and that the utility function is of constant relative risk aversion form, with a coefficient of relative risk aversion equal to γ.
(a) Derive and interpret the modified golden rule condition in this case.
(b) Characterize the dynamics of consumption and capital accumulation.
(c) Suppose that the economy is in steady state and that ϕ decreases permanently and unexpectedly. Describe the dynamic adjustment of the economy to this adverse supply shock.

3. *Optimal consumption with exponential utility.*

Consider a family, growing at rate n and with discount rate θ, that faces a given path of future wages and interest rates and has a constant absolute risk aversion utility function, with a coefficient of risk aversion α. Solve for the path of consumption, as is done in the text for the CRRA utility function.

4. *Government spending in the Ramsey model.*

(a) In the Ramsey model, suppose that the government unexpectedly increases government spending, raising it from a base level g_0 to the level g_1 (per capita in both cases), starting from steady state. Analyze the effects of this increase on the paths of consumption and capital accumulation.
Note: You may want to use the equivalence between the command and market solutions and treat the increase in g as a negative additive productivity shock.
(b) Do the same exercise, assuming that the economy is not initially in steady state. Characterize the dynamic effects when utility is of the CARA form. Explain.
(c) Suppose, instead, that the increase in government spending is announced at time t_0 to take place at time t_1, with $t_1 > t_0$. Characterize the dynamic effects on consumption and capital accumulation from t_0.
Note: Phase diagrams are convenient to use when characterizing the effects of such anticipated changes. Note that between t_0 and t_1 the equations of motion are given by the dynamic system with $g = g_0$, and that after t_1 the equations of motion are given by the dynamic system with $g = g_1$. Note further that c cannot jump anticipatedly at time t_1. Note finally that k at time t_0 is given and that the system must converge to the new equilibrium. Show that these conditions uniquely define the path of adjustment. (Abel 1981 characterizes the effects of anticipated or

temporary changes in taxation on investment within the q theory using such phase diagrams.)

5. *Savings and investment with costs of adjustment in a closed economy. (This follows Abel and Blanchard 1983.)*

Assume that there are costs of adjusting the capital stock, as in section 2.4, but that the economy is closed. Derive the optimal paths of consumption and capital accumulation in this case and provide an explanation of the difference between the Euler equation for this case and equation (7).

6. *Foreign debt and trade surpluses.*

(a) Using the relevant budget constraint, show that b_0, the initial value of external debt, is equal to the present value of net exports, provided an NPG condition is satisfied.
(b) Suppose that for some period of time a country's external debt is growing more rapidly than at the rate $r - n$. What can you conclude about the likelihood that the NPG condition will be violated in the long run? What then is the relevance of the NPG condition?

7. Suppose that in a closed economy there is an unexpected permanent reduction in the efficiency of production, represented in the symbols in the text as an increase in z_0. Assuming that the economy started in a steady state, derive and explain its optimal dynamic adjustment toward the new steady state.

8. *Growth with increasing returns, I.*

Consider an economy with the production function

$$Y = K^{a+b}N^{1-a}, \qquad b > 0,\ a + b < 1$$

so that there are increasing returns to scale but decreasing returns to capital given labor. Population is growing at the rate n, and there is no depreciation.
(a) Show that it is possible for capital, output, and consumption all to grow at the same rate g. This is known as *balanced growth*. Derive the balanced growth rate g, and explain its dependence on a, b, and n.
(b) Suppose that the felicity function for the representative family is

$$u(c_t) = \ln c_t$$

and that the family has a constant discount rate θ.

Assuming that the economy converges to a balanced growth path, characterize the steady state marginal product of capital. Compare it to the modified golden rule level that would obtain under constant returns (i.e., with $b = 0$). Explain the difference.

9. *Growth with increasing returns, II. (This follows Rebelo 1987.)*

Consider the following economy: Population is constant and normalized to unity, and the representative individual maximizes

$$\int_0^\infty U(C)\exp(-\theta t)\,dt.$$

K is the capital stock in the economy and can be used either to produce consumption goods or new capital goods. Let x, $0 \leqslant x \leqslant 1$, be the proportion of capital used in the production of consumption goods. The two production functions for consumption and investment goods are given by

$$C = F(xK),$$

$$F(0) = 0, \qquad F'(\cdot) > 0,\ F''(\cdot) < 0.$$

$dK/dt = I = B(1 - x)K$; B is a positive constant. Capital does not depreciate.
(a) What is the maximum growth rate of capital in this economy? What is the associated level of consumption?
(b) Derive the first-order conditions associated with this maximization problem. Interpret them. Give, in particular, an interpretation of the Lagrange multipliers and costate variables as shadow prices.
(c) Assume that $F(xK) = A(xK)^a$, where $0 < a < 1$, and that $U(C) = \ln(C)$. Show that if the economy converges to a balanced growth path, the rate of growth of consumption is given by $a(B - \theta)$. Explain in words.

What happens to the relative price of capital goods in terms of consumption goods along the balanced growth path?
(d) Contrast your results with those obtained in the conventional Ramsey model. Explain why they differ.
(e) How does this model do in terms of explaining the basic facts of growth as laid out by Kaldor and Solow, and summarized in chapter 1? What is the relation of consumption to income along the balanced growth path? What is the relation of output to capital? (Be careful about how you define capital—value or volume—here.)

Notes

1. In chapter 3 we show that people who have finite lives may still act as if they in effect had infinite lives.

2. Frank Ramsey was a Cambridge, England, mathematician and logician who died at the age of 26. His genius is evidenced by the fact that he had written three classic articles in economics by the age at which many economists are contemplating leaving graduate school. J. M. Keynes (1930) eulogizes Ramsey.

3. If depreciation is exponential at the rate λ, then gross output is $Y + \lambda K = F(K, N) + \lambda K = G(K, N)$. If $F(K, N)$ is degree one homogeneous, so is $G(K, N)$.

4. An alternative plausible formulation is the so-called Benthamite welfare function in which the felicity function becomes $N_t u(c_t)$ so that the number of family members receiving the given utility level is taken into account. Recognizing that $N_t = N_0 e^{nt}$, we see that the Benthamite formulation is equivalent to reducing the rate of time preference to $(\theta - n)$ because the larger size of the family at later dates in effect

increases the weight given to the utility of the representative individual in a later generation.

In assuming that $\theta > 0$, we depart from Ramsey who, interpreting the maximization problem as the problem solved by a central planner, argued that there was no ethical case for discounting the future.

5. Ordinary calculus optimization methods have to be augmented to handle the presence of a time derivative in constraint (2). Intriligator (1971) provides an introduction to intertemporal optimization methods.

6. A warning is in order here. First, under weaker assumptions than those made in the text, for example, a linear production function or no discounting, an optimum may not exist. Even if an optimum does exist, the transversality condition, equation (8), may not be necessary. But if one is ready to set sufficiently strong conditions for the maximization problem, these problems can usually safely be ignored. For a more careful statement and further discussion, see Shell (1969) and Benveniste and Scheinkman (1982).

7. Note from the formulation of the central planner's problem that it is implicitly assumed that capital can be consumed.

8. We emphasize again that as intuitive as this argument for the transversality condition is, there are infinite horizon problems in which the transversality condition is not necessary for the optimal path. See Shell (1969) and Michel (1982).

9. To show that the utility function converges to the logarithmic function as γ tends to unity, use L'Hospital's rule.

10. On the basic measures of risk aversion, see J. Pratt in Diamond and Rothschild (1978); see also the following articles in Diamond and Rothschild by Yaari and by Rothschild and Stiglitz.

Behavior toward risk and the degree of substitution between consumption at different times are conceptually two different issues. Under the assumption that the von Neumann-Morgenstern utility integral is additively separable over time, however, the two depend only on the curvature of the instantaneous utility function and are thus directly related. See chapter 6 for further discussion.

11. In steady state, with $dk/dt = 0$, we have from (2),

$$c^* = f(k^*) - nk^*.$$

Maximization of c^* with respect to k^* gives the golden rule, that the marginal product of capital (or interest rate) is equal to the growth rate of population.

12. We freely interchange the marginal product and interest rates. We show later that in the decentralized Ramsey economy, the two are indeed equal.

13. The result that the steady state interest rate does not depend on the utility function can, however, be easily overturned. If labor-augmenting (Harrod-neutral) technical progress is taking place at the rate μ, so that

$$Y_t = F[K_t, \exp(\mu t)N_t]$$

and if the utility function is of the CRRA class, then the modified golden rule condition becomes $f'(k^*) = \theta + \sigma\mu + n$. [In this case k^* is the ratio of capital to effective labor, i.e., $K_t/\exp(\mu t)N_t$, and the steady state is one in which consumption per capita is growing at the rate μ.]

14. The analysis can also be undertaken in (k, λ) space, using the first-order condition (6).

15. The behavior of consumption on the horizontal axis, where $c = 0$, depends on the value of the instantaneous elasticity of substitution $\sigma(c)$ for $c = 0$. Equation (7″) implies that

$$\frac{dc}{dt} = \sigma(c)[f'(k) - \theta - n]c.$$

If $\sigma^{-1}(0)$ is not zero, then $dc/dt = 0$ when $c = 0$. We assume this to be the case. If the condition is not met, one must examine the behavior of $c\sigma(c)$ at $c = 0$.

16. Throughout the book we will encounter phase diagrams in which there is only one convergent path. Although we will often simply assume that the economy proceeds on this converging path, an argument must be made in each case that the converging path is the only one that satisfies the conditions of the problem. As we will see in chapter 5, there are cases in which we cannot rule out some of the diverging paths.

17. Changes in f'' and θ affect both the rate of convergence to the steady state and the steady state capital stock itself.

18. The condition that the rental rate on capital is equal to the interest rate is special to this one-good model. If the relative price of capital, p_k, could vary, asset market equilibrium would ensure that the expected rate of return from holding capital would be equal to the interest rate. The rate of return from holding capital is the rental rate, r_k plus any capital gains on capital minus depreciation, all expressed relative to the price of the capital:

$$\text{rate of return} = \frac{r_k + (dp_k/dt) - \delta p_k}{p_k} = \text{real interest rate},$$

where δ is the rate of depreciation. In the single-good model, p_k is identically one, so there are no changes in the relative price of capital, and we are assuming that δ is zero; accordingly, the rate of return on capital is r_k, which is equal to the interest rate. (We are implicitly assuming that the economy never specializes completely; if it did not save at all, the relative price of capital goods could be less than one; if it did not consume at all, the relative price of capital could exceed one.)

19. For notational convenience we shall assume that there is just one family and one firm, both acting competitively.

20. There are many alternative ways of describing the decentralized economy. For example, firms can own the capital and finance investment by either borrowing or issuing equity. Or, instead of operating with spot factor markets, the economy may

operate in the Arrow-Debreu complete market framework in which markets for current and all future commodities, including services, are open at the beginning of time; all contracts are made then, and the rest of history merely executes these contracts. Under perfect foresight, all these economies will have the same allocation of resources.

21. We limit ourselves in what follows to paths of wages and rental rates such that the following condition is satisfied:

$$\lim_{t\to\infty} \exp\left[-\int_0^t (r_v - n)\,dv\right] = 0.$$

This condition says, roughly, that asymptotically the interest rate must exceed the rate of population growth. We will show that the equilibrium path indeed satisfies this condition. A complete argument would show that if this condition is not satisfied, there is no equilibrium. See note 25 below for further elaboration.

22. In the present model, in which all families are the same, they will in equilibrium have the same wealth position and hold the same fraction of the capital stock. Since the aggregate capital stock must be positive, each family will, in equilibrium, have positive wealth. This is, however, a characteristic of equilibrium, not a constraint that should be imposed a priori on the maximization problem of each family. In an economy with heterogeneous families, or families with different paths of labor income, positive aggregate capital may coexist with temporary borrowing by some families.

23. Charles Ponzi, one of Boston's sons, made a quick fortune in the 1920s using chain letters. He was sent to prison and died poor.

24. This raises the question of how the no-Ponzi-game condition is actually enforced. The fact that parents cannot, for the most part, leave negative bequests to their children implies that family debt cannot increase exponentially. It may in fact impose a stronger restriction on borrowing than the no-Ponzi-game condition used here.

25. Following up on note 21, there is one loose end in our proof of equivalence, which we now tie up. We have restricted ourselves to paths where the interest rate exceeds asymptotically the population growth rate. Given this restriction, we showed that there is an equilibrium path, which is the same as the central planning one, so that r converges asymptotically to $n + \theta$. We now need to show that paths on which the interest rate is asymptotically less than n, cannot be equilibria. To see why, rewrite the budget constraint facing the family as

$$\frac{da_t}{dt} = (r_t - n)a_t + (c_t - w_t).$$

Consider then two paths of consumption, which have the same level of consumption after some time T, so that $c_t - w_t$ is the same on both paths after T. Then, if $r_t - n$ is asymptotically negative, both paths will lead to the same asymptotic value of a (the same level of net indebtedness if a is negative). If one path satisfies the no-Ponzi-game condition, so will the other. But this implies that the family will always want to have very high (possibly infinite) consumption until time T. This cannot be an equilibrium.

26. We consider endogenous government spending in chapter 11.

27. Government spending, for instance, on education, might substitute for private spending, in which case the utility function would have to be amended appropriately. Similarly, government spending on defense and public safety might contribute to the economy's productive capacity, but we do not model any such effects.

28. The dynamics of investment and savings in a closed economy with adjustment costs are studied in Abel and Blanchard (1983).

29. Blanchard (1983), Fischer and Frenkel (1972), and Svensson (1984) have used similar models to examine the dynamics of foreign debt and the current account.

30. Investment decisions based on adjustment costs have been modeled by Abel (1981), Eisner and Strotz (1963), Lucas (1967), and Tobin (1969). Our specification is that of Hayashi (1982).

31. The conditions specified after equation (31) ensure the properties of the installation cost function $iT(i/k)$. Note that, in practice, when capital depreciates, the costs of small rates of disinvestment, which can take place through depreciation, are likely to be very small or zero.

Instead of defining both a production and an installation cost function, we could have defined a 'net' production function that gives output available for consumption or export, $H(K, N, I)$. This is the approach taken, for example, by Lucas (1967). In our case $H(K, N, I) = F(K, N) - I[1 + T(I/K)]$, where uppercase letters are total amounts of corresponding per capita variables. The function $H(\cdot)$ has constant returns to scale if $F(\cdot)$ does.

32. If the world interest rate had differed from the rate of time preference, the country would either accumulate or decumulate forever. This follows from the Euler equation in the absence of population growth, which from section 2.2 will give $[du'(c_t)/dt]/u'(c_t) = \theta - r$, where r is the interest rate. If the country accumulates forever because $\theta < r$, then it eventually becomes a large economy and begins to affect the world interest rate; if $\theta > r$, then the country runs its wealth down as far as it can. To avoid these difficulties, we set $\theta = r$. We could also obtain convergence to a steady state if we specified a time path for the world interest rate that converges to θ, rather than always being equal to θ. We assume $r = \theta$ for simplicity.

33. We state the NPG condition as an equality. We could again state it as an inequality, requiring the present discounted value of debt to be nonnegative. But if marginal utility is positive, the central planner will not want to accumulate increasing claims on the rest of the world forever. Thus the NPG condition will hold with equality.

34. Defining the costate variable on (31) as $\mu_t q_t \exp(-\theta t)$ rather than as a single variable is a matter of convenience, as will become clear later when we show that q plays a key role in determining investment.

35. Note that because of the equality of the interest rate and the subjective discount rate, the marginal propensity to consume out of wealth is equal to θ independently of the form of the felicity function.

36. Note that given constant μ_t, equation (39) implies that $\lim q_t k_t \exp(-\theta t) = 0$ as t goes to ∞. This is, however, not the same as $\lim q_t \exp(-\theta t) = 0$ as t goes to ∞, which is the condition needed to derive (44). To derive (44), one must characterize the phase diagram associated with equations (37′) and (43) and show that the only path that satifies these equations and the transversality condition (39) is a path where both k and q tend to k^* and q^*, respectively, so that $\lim q_t \exp(-\theta t) = 0$ as t goes to ∞.

37. This way of thinking about the investment decision was developed by Tobin. For that reason, q is often called Tobin's q. See Hayashi (1982) for a discussion of the relation of q to its empirical counterparts; in particular, Hayashi discusses the conditions under which average q, as reflected, say, in the stock market valuation of a firm, is equal to marginal q, the shadow value of an additional unit of installed capital. Marginal and average q are equal, leaving aside tax issues, if the firm's production function and the adjustment cost function $iT(\cdot)$ are each first-degree homogeneous and firms operate in competitive markets. Under those assumptions one would expect a tight relation between the market valuation of firms and their investment decisions. Empirically, although average q and investment rates are indeed correlated, the relation is far from tight (see Hayashi 1982).

38. If there is population growth at the rate n, then q^* is given by $n = \phi(q^*)$ so that $q^* > 1$, and k^* is given by

$$\theta q^* = f'(k^*) - n^2 T'(n).$$

39. The restriction to local dynamics ensures that $dq/dt = 0$ is negatively sloped; away from the steady state there is no assurance that the slope of $dq/dt = 0$ is negative without imposing more conditions on the $T(\cdot)$ function. However, the restrictions imposed on $T(\cdot)$ are sufficient to ensure that there is a unique steady state in the neighborhood of which the $dq/dt = 0$ locus is negatively sloped.

40. In appendix D we show that the transversality condition suffices in the linearized system to rule out any divergent paths that satisfy the necessary conditions (47).

41. The current account always has present discounted value equal to zero when condition (32) is satisfied; it is only when the initial debt is zero that the same applies to the trade account.

42. Given the equivalence between the command optimum and the decentralized economy, the shocks can also be interpreted as taxes, where the government is using the proceeds of the taxes to finance government spending that does not affect the utility function, as in section 2.3.

43. This experiment raises the methodological issue of how unexpected changes can occur in a model in which there is perfect foresight. The correct way to analyze such changes would be to set up the maximizing problems of the central planner or economic agents explicitly as decision problems under uncertainty. This substantially complicates the analysis, and we defer this to chapter 6; we can think of the approach taken here as a shortcut in which the surprise is an event that was regarded as so unlikely as not to be taken into account up to the time it occurs.

44. If individuals dislike changes in the rate of consumption so that the felicity function is, for instance, $u(c, dc/dt)$, the reduction in z_1 would cause a smaller decline in consumption than in output initially; the country would in that case initially borrow abroad temporarily to cushion the shock of the reduction in the standard of living, and end up with permanently higher debt and lower consumption.

45. We briefly return in chapter 7 to the issue of nonseparability in the context of a discussion of labor supply.

As we shall see, however, the distinction between the felicity function and the discount factor becomes somewhat blurred when we allow for more general formulations of this discount factor.

46. The argument to this point does not eliminate the possibility that there is no steady state. The argument of this paragraph can be seen, however, to imply the existence of a steady state with $r = \theta_m$.

47. The no-Ponzi-game condition prevents the shortsighted from going further and further into debt.

48. Ramsey (1928) conjectured this result; it was proved by Becker (1980).

49. Note the similarity between the discussion here and that of the relationship between the world interest rate and rate of time preference of a small country in section 2.4.

50. Because the point we are about to make about the optimal program does not depend on the presence of $x(t)$ in the discount function, we omit that argument henceforth.

51. This result is due to Strotz (1956).

52. An example is $D(\cdot) = \max[0, A - \theta(t - s)]$.

53. See Elster (1979) and Schelling (1984) for more extensive discussion of how people do and should deal with inconsistencies. Issues of time consistency also arise in the context of games between agents or between agents and the government. We will study these in chapter 11.

54. There is no "correct" way to behave when tastes are dynamically inconsistent, for there is no way of knowing which is the right set of tastes: the title "Ulysses and the Sirens" (Elster 1979) refers to Ulysses's strategy of having himself tied to the mast to avoid succumbing to the Sirens' cry—but maybe the real Ulysses was the one who would have succumbed if the other Ulysses hadn't tied him to the mast.

55. Epstein and Hynes (1983) suggest an alternative specification, namely,

$$\int_0^t \exp\left[-\int_0^s u(c_v)\,dv\right] ds.$$

This specification has the same qualitative implication as Uzawa's but is more tractable analytically. Note that in this form there is no longer any distinction between the discount rate and the instantaneous felicity function.

56. Lucas and Stokey (1984) work with a model of this type.

57. Once again, there are many alternative ways of describing the decentralized economy. Firms could, instead, finance investment by issuing shares or by borrowing either abroad or domestically. The real allocation would be the same in all cases.

58. If investment is so high that net cash flows are negative, the firm is, in effect, issuing equity by paying a negative dividend, that is, making a call on stockholders for cash.

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3

The Overlapping Generations Model

The overlapping generations model of Allais (1947), Samuelson (1958), and Diamond (1965) is the second basic model used in micro-based macroeconomics. The name implies the structure: at any one time individuals of different generations are alive and may be trading with one another, each generation trades with different generations in different periods of its life, and there are generations yet unborn whose preferences may not be registered in current market transactions.

The model is widely used because it makes it possible to study the aggregate implications of life-cycle saving by individuals. The capital stock is generated by individuals who save during their working lives to finance their consumption during retirement. The determinants of the aggregate capital stock as well as the effects of government policy on the capital stock and the welfare of different generations are easily studied. The model can be extended to allow for bequests, both intentional and unintentional.

Given the descriptive appeal of the life-cycle hypothesis, these uses of the model would alone justify its widespread popularity. But beyond that, the model provides an example of an economy in which the competitive equilibrium is not necessarily that which would be chosen by a central planner. There is an even stronger result: the competitive equilibrium may not be Pareto optimal. Life-cycle savers may overaccumulate capital, leading o equilibria in which everyone can be made better off by consuming part f the capital stock. This possible inefficiency of the equilibrium contrasts harply with the intertemporal efficiency of the competitive equilibrium in he Ramsey model. One of the goals of this chapter is to elucidate the aspects f the life-cycle model that make inefficiency possible.

We start, in section 3.1, with the simplest version of the overlapping enerations model in which individuals live for only two periods. Starting om individual maximization, we show how the aggregate capital stock olves over time. We extend the model to consider the effects of altruism