Final Exam ECGA7020 D. McLeod Answer Key Fall 2005

Everyone must answer questions 1-3 and question 4 or 5 to the best of your ability. Only after you have completed these four questions answer one of the remaining questions for extra credit. Answer at most 5 questions. Total points on this exam: 60 points.

1. Simple RBC model (16 points)
2. Precautionary savings. (16 points)
3. Choosing a Central Banker (16 points)
4. Portfolio choice (12 points)
5. Asset pricing (12 points)
6. *Ramsey model with taxes (6 points)*

1. An RBC model with stochastic productivity shocks to R&D *(borrowed from the 2001 UCB final exam for Econ 202B--Romer's macro course)***:** Income Y_t is produced with labor and technology, which has a stochastic component and depends on investment in R&D (analogous to investment in capital). Output depends on $Y_t = A_t^{\alpha} L_t$, where $0<\alpha<1$ $A_{t+1} = R_t z_{t+1}$ *and*, $Y_t = C_t + R_t$ Where z_{t+1} is a random shock to the productivity of investment in research R_t Let $\log(z_t)$ follows the following process: $\log(z_t) = \eta_t + \theta\eta_{t-1}$, where $\eta_t \sim N(0,\sigma^2)$. The preferences are given by

$E_t\left[\sum_{t=0}^{\infty}\beta^t u(C_t, L_t)\right]$, where $0<\beta<1$.

a) Using the social planner's Bellman equation,

$$V(A_t, z_t) = \max\left\{u(C_t, L_t) + \beta E_t\left[V(A_{t+1}, z_{t+1})\right]\right\} \quad s.t. \quad A_{t+1} = [A_t^{\alpha} L_t - C_t]z_{t+1},$$

Distinguish between state and control variables and derive the first order conditions. Using the envelope theorem derive the IT Euler equation for consumption. The control variables are C_t and L_t and the state variable are A_t and z_t. The Bellman Equation is,

$$V(A_t, z_t) = \max_{C_t, L_t}\left\{u(C_t, L_t) + \beta E_t\left[V(A_{t+1}, z_{t+1})\right]\right\}$$

$$s.t.: \quad A_{t+1} = [A_t^{\alpha} L_t - C_t]z_{t+1}, \quad \log(z_t) = \eta_t + \theta\eta_{t-1}. \tag{1.1}$$

First order conditions for consumption and labor (leisure):

$$\begin{aligned} &[c_t :]\ u_c'(C_t, L_t) - \beta E_t\left[V'(A_{t+1}, z_{t+1})z_{t+1}\right] = 0 \\ &[L_t :]\ u_L'(C_t, L_t) + \beta E_t\left[V'(A_{t+1}, z_{t+1})A_t^{\alpha} z_{t+1}\right] = 0 \end{aligned} \tag{1.2}$$

From (1.2) we have: $$\frac{u_L'(C_t, L_t)}{u_c'(C_t, L_t)} = -A_t^{\alpha} = \frac{\partial Y_t}{\partial L_t} \tag{1.3}$$

Using the envelope theorem derive the IT Euler equation for consumption: Applying the envelope theorem**,

$$\frac{\partial V(A_t,z_t)}{\partial A_t} = \beta E_t\left[\frac{\partial V(A_{t+1},z_{t+1})}{\partial A_{t+1}}\frac{\partial A_{t+1}}{\partial A_t}\right] \quad \text{where} \quad A_{t+1} = [A_t^{\alpha}L_t - C_t]z_{t+1},$$

$$\begin{aligned} V_A'(A_t,z_t) &= \beta E_t\left[V'(A_{t+1},z_{t+1})\alpha A_t^{\alpha-1}L_t z_{t+1}\right] \\ V_A'(A_t,z_t) &= \beta\alpha A_t^{\alpha-1}L_t E_t\left[V'(A_t,z_{t+1})z_{t+1}\right] \end{aligned} \qquad (1.4)$$

But note that the f.o.c. for consumption (1.2) implies, $u_c'(C_t,L_t) = \beta E_t\left[V'(A_{t+1},z_{t+1})z_{t+1}\right]$ so that,

$$V_A'(A_t,z_t) = \alpha A_t^{\alpha-1}L_t u_c'(C_t,L_t) \qquad (1.5)$$

Iterating (1.5) forward one period,

$$V_A'(A_{t+1},z_{t+1}) = \left[\alpha A_{t+1}^{\alpha-1}L_{t+1}\right]u_c'(C_{t+1},L_{t+1}) \qquad \mathbf{(1.6)}$$

Substituting (1.6) into $u_c'(C_t,L_t) - \beta E_t\left[V'(A_{t+1},z_{t+1})z_{t+1}\right] = 0$, yields the IT Euler equation equating consumption driven changes in utility today to expected changes in utility in t+1:

$$\begin{aligned} u_c'(C_t,L_t) &= \beta E_t\left[\left[\alpha A_{t+1}^{\alpha-1}L_{t+1}\right]u_c'(C_{t+1},L_{t+1})z_{t+1}\right] \\ u_c'(C_t,L_t) &= \beta E_t\left[u_c'(C_{t+1},L_{t+1})\alpha\frac{Y_{t+1}}{A_{t+1}}z_{t+1}\right] \end{aligned} \qquad \mathbf{(1.7)}$$

Unfortunately, Romer uses the "direct attack" method to derive the solution to a very similar RBC model in problem 4.11. What is the advantage of the above solution over the direct attack method? (Apart from whether the algebra is easier) For solutions using the envelope theorem see eqs 1.8-1.9 of the handout, **Formal solution to Romer Problem 4.11 or see equation 2.13 page 34 of Drazen Chapt. 2 pp. 31-35 or equation 9 of Obstfeld and Rogoff's Supplement to chapter 2 or page 13 of Adda and Cooper Chapter 2 and/or especially the formal solution to Blanchard and Fisher's Chapt 6 optimal consumption model class handout – all of these links to the course web page should work if you are connected to internet as you view this file. Of course since the log utility function is of the HARA class, thanks to Merton(1971) we know the functional form of value function is logarithmic as well, a fact that Romer 's solution guide author takes advantage of (and later confirms) when he "guesses" the value function shown in equation (9) of that guide.

b. Assume that the utility is given by $u(C_t, L_t) = \log(C_t) + b\log(1-L_t)$ for the remainder of the problem, determine the share of output devoted to R&D, $s = \frac{R}{Y}$., at any point in time:

Since $u_c'(C_t, L_t) = \frac{1}{C_t}$ using (1.7), we have:

$$\frac{1}{C_t} = \beta E_t\left[\frac{1}{C_{t+1}}\alpha\frac{Y_{t+1}}{A_{t+1}}z_{t+1}\right] \qquad \textbf{(1.8)}$$

We know $A_{t+1} = [Y_t - C_t]z_{t+1} = sY_t z_{t+1}$, we can replace A_{t+1} in the bracketed part of (1.8):

$$\frac{1}{C_t} = \beta E_t\left[\frac{1}{C_{t+1}}\alpha\frac{Y_{t+1}}{sY_t z_{t+1}}z_{t+1}\right]$$
$$\frac{1}{(1-s)Y_t} = \beta E_t\left[\frac{1}{(1-s)Y_{t+1}}\alpha\frac{Y_{t+1}}{sY_t z_{t+1}}z_{t+1}\right] \qquad \textbf{(1.9)}$$

Note that all the expected t +1 terms inside the term in brackets cancel, so that multiplying both sides of 1.9 by $(1-s)Y_t$ yields $s = \alpha\beta$. The answer to this part of the problem follows the class handout **Formal solution to Romer Problem 4.11** very closely. In fact the solution is identical except that Romer's writes $\beta = e^{-\rho}$ so that in our notation $C_t/Y_t = (1 - \alpha\beta)$ (see the answer to problem 4.11 equation (26)).

c) Solve for L_t : using the combined the first order terms shown in (1.3)

$$\frac{u_L'(C_t, L_t)}{u_c'(C_t, L_t)} = -A_t^{\alpha} = \frac{\partial Y_t}{\partial L_t} \quad \textit{which implies,} \quad \frac{\frac{-b}{1-L_t}}{\frac{1}{C_t}} = \frac{-bC_t}{1-L_t} = -A_t^{\alpha}$$

$$\frac{C_t}{1-L_t} = \frac{A_t^{\alpha}}{b} = \frac{Y_t}{bL_t} \quad \textit{replacing } C_t \textit{ with } (1-s)Y_t$$

$$\frac{(1-s)Y_t}{1-L_t} = \frac{Y_t}{bL_t} \quad \textit{cancel } Y_t \textit{ so that } \frac{(1-s)}{1-L_t} = \frac{1}{bL_t} \quad \textit{and solve for } L_t, \; bL_t - sbL_t = 1 - L_t$$

$$L_t = \frac{1}{1+b(1-\alpha\beta)} = L^*$$

Where the last (constant) expression for L* makes use of the fact that $s = s^* = \alpha\beta$ in the steady state. Note that this solution is slightly less complex that Romer's (also constant) labor supply equation (27)—why?

b) Assume that the utility is given by $u(C_t, L_t) = \log(C_t) + \phi \log(1 - L_t)$ for the remainder of the problem. Determine the share of output devoted to R&D, $s = \frac{R}{Y}$., at any point in time.

c) Solve for L_t at each point in time, discussing how shocks to R&D affect employment and wages.

2. Precautionary Savings: Assume consumers maximize, $E_t\left[\sum_{t=0}^{\infty} \beta^t u(C_t)\right]$

s.t. $c_t + B_{t+1} + k_{t+1} = W_t$ and $Wt = \tilde{A}_t k_t^{\alpha} + (1+r)B_t$ *where* $\tilde{A}_t$ *is i.i.d.* $(\bar{A}, \sigma^2)$

using the HARA utility function $U(c_t) = -\frac{1}{\phi}\exp(-\phi c_t)$

Where c_t and k_t are consumption and capital per capita, while B_t is a risk free asset paying certain return $R_t = (1 + r)$ and population growth n = 0.

a) Set up the Bellman equation for the consumer's maximization problem. Identify the state and control variables.

b) Is precautionary savings likely to be positive in this model, given that income from domestic capital is not diversifiable? Japelli and Pagano (1994) find OECD savings falls when credit markets develop further, arguing that this reflects lower precautionary savings? What is their reasoning, can this same argument explain the recent decline U.S. household savings?

c) Derive the first order condition and sketch a strategy for solving the Bellman explicitly for the path on consumption recall the log normal distribution is,

$$E_t[\exp(x)] = \exp\left\{E(x) + \frac{\sigma_x^2}{2}\right\}$$

Hint: Use Adda and Cooper's direct attack method, assuming a functional form for the value function and literally taking the expectation of $V(W_{t+1})$ – you may be able to make some progress using an envelope theorem, but this may be one of those problem where the envelope theorem is not helpful.

d) (more difficult) If you have time, solve explicitly for the time path of c_t and identify the term reflecting precautionary savings, that is "rainy day" savings in response to risk.

3. Choosing the right Central Banker: In the Kydland-Prescott/Barro-Gordon framework the Central bank faces a large number of private agents seeking to predict inflation or minimize, $MinE_{t-1}(\pi_t - \pi_t^e)^2$

Luckily, the Central Bank (CB) sets π_t to minimize its loss function

$L = \min\left\{(y_t - y^*)^2 + a(\pi_t)^2\right\}$ subject to the Lucas supply curve: $y_t = \bar{y} + b(\pi_t - \pi_t^e) - z_t$

after observing the a supply shock z_t and the private sector's best inflation forecast, π^e. The CB's targets for output, y*, is a bit ambitious in that. $y^* > \bar{y}$. Alternatively, we can think of a distortion or wedge $y^* - \bar{y} = k$ between desired and average output resulting from monopolistic market power or taxes, denoting this gap as k (as in Obstfeld and Rogoff section 9.5).

a) Solve for an equilibrium inflation rate π_t and π_t^e keeping in mind that the $z_t \sim N(0,\sigma^2)$ and simplifying by setting b = 1. Show the equilibrium y_t and the amount of output stabilization that takes place under this regime, given the Lucas supply curve. Use the graph plotting π_t against π_t^e to show how a more conservative central banker a' > a affects both the equilibrium inflation rate and the amount of output stabilization. Suppose the L(a) and not L(a') is society's loss function (SWF). Is society always better off appointing a conservative central banker? What is the tradeoff society faces (hint: compute the variance of y_t with and w/o a conservative central banker)?

b) One way for society to get lower inflation rate is to offer the CB head a bonus which declines with inflation at rate ω,

$$L_1 = \min\left\{(y_t - y^*)^2 + a(\pi_t)^2 + 2\omega * \pi_t\right\}.$$

$$L_2 = \min\left\{(y_t - y^*)^2 + a(\pi_t)^2 + 2\omega z_t \pi_t\right\}$$

Compare the two bonus schemes, the optimal bonus in the first scheme is ω= k, why? What is the advantage of the second scheme where the bonus is linked to the supply shock z_t particularly when one is dealing with a conservative central banker? Rogoff (2004) argues globalization has made the economy more competitive, reducing k. How does a reduction in k affect long term inflation and stabilization (the variance of y_t).

c) Suppose a new CB head introduces an inflation target π* > 0 modifying the CB loss function above to,

$$L = \min\left\{(y_t - y^*)^2 + a(\pi_t - \pi^*)^2\right\} \text{ compute the new } \pi_t \text{ and } \pi_t^e.$$

What are the consequences of raising π* for output stabilization and expected inflation. Is there any benefit from raising target inflation from 0 to π*? Suppose proposed Fed Chair Bernanke's was an anti-deflation hawk: why would he be happier with π* > 0 as his inflation target?

4. **Asset Pricing:** The economy consists of identical infinitely-lived consumers, the representative consumer maximizes

$$E_t\left[\sum_{t=0}^{\infty}\beta^t u(C_t)\right].$$

Subject to $W_{t+1} = R_t[W_t + Y_t - C_t]$, where R_t is given by $R_t = \frac{p_{t+1} + d_{t+1}}{p_t}$. Let p_t to be the current price of an asset, and d_{t+1} the dividends received by holding the asset from t to t+1 so that the return of this investment is given R_t. Let W_t the current financial wealth of the consumer.

a) Using Dynamic programming, set up the Bellman Equation
b) Derive the first order condition and the envelope theorem.
c) Derive the Basic Pricing Equation $p_t = E_t\left[\beta\left(\frac{u'(C_{t+1})}{u'(C_t)}\right)[p_{t+1} + d_{t+1}]\right]$, and assuming that $C_t = d_t$ (Competitive equilibrium and considering only one unit of asset), state the Lucas Asset Pricing Equation.
d) Solving the Basic Pricing Equation by forward yields $p_t = E_t\left[\beta\left(\frac{u'(C_{t+1})}{u'(C_t)}\right)d_{t+1}\right]$

Assuming log utility, $u(C_t) = \log(C_t)$, derive the asset price p_t. Explain the intuition and novelty of

this asset pricing equation. Is the log utility function plausible given the empirical evidence on consumption and savings? In what sense is this asset pricing formula analogous to a certainty equivalent price, and to q_t with quadratic adjustment costs? What makes this asset pricing model economics and not finance?

5. Portfolio Choice: Newly minted PhD chooses to save in one or both of two assets, one risky and one safe. With no initial wealth, all savings must be financed out of current income and can save current income through these two assets. The safe risk free non-stochastic gross return is R^S. The risky asset $\tilde{R}^r$ and a mean return of $\bar{R}^r$. Let a^r and a^s are holdings of asset type j = r, s since there is only the present and future. Assets' prices are normalized to 1 in period 1.

The portfolio choice problem can then be written as

$$\max_{a^r, a^s} u(y_1 - a^r - a^s) + \beta E_1\left[u(\tilde{R}^r a^r + R^s a^s + y_2)\right]$$

1. Assuming that y_1 is known with certainty. Derive the first-order conditions.

2. Under which conditions the consumer wants to hold both types of assets, or only one asset.
3. What are the implications if the average return on both assets is equal?
4. What is the arbitrage condition between risky asset and non-stochastic asset?