Formal DP Solution to Romer's Baseline RBC Model

Technology: Production function:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \qquad 0 < \alpha < 1 \tag{1.1}$$

Capital accumulation:

$$K_{t+1} = (1-\delta)K_t + I_t, \qquad 0 < \delta \leq 1, K_0 \geq 0 \tag{1.2}$$

National accounts:

$$Y_t = C_t + I_t + G_t \tag{1.3}$$

Population:

$$L_t = 1, \qquad \text{for all } t \tag{1.4}$$

Technology shocks:

$$\log A_t = \log \bar{A} + gt + \tilde{A}_{t,} \text{ where } \tilde{A}_t = \rho\tilde{A}_{t-1} + \varepsilon_{t,} \text{ with } |\rho|<1 \text{ and } \varepsilon_t \sim \eta[0,1] \tag{1.5}$$

Government expenditure shock:

$$\log G_t = \log \bar{G} + gt + \tilde{G}_{t,} \textit{ where } \tilde{G}_t = \rho_G \tilde{G}_{t-1} + \eta_{t,} \textit{ with } |\rho_G|<1 \text{ and } \eta_t \sim \eta[0,1] \tag{1.6}$$

1. Preferences

$$U_0 \equiv E_0 \sum_{t=0}^{\infty} \beta^t u[C_t, 1-l_t] \text{ with } u[C_t, 1-l_t] = \log C + b\ln(1-l) \tag{1.7}$$

2. Information

$$\varepsilon_{t-s} \textit{ and } \eta_{t\text{-s}} \text{ are known at time t ; given } \varepsilon_{-1} \textit{ and } \eta_{\text{-1}}. \tag{1.8}$$

Competitive Equilibrium

A competitive equilibrium in an RBC economy is a stochastic process for the collection

$$\{C_t, l_t, V_t, K_t, L_t, A_t, G_t, Y_t, I_t\}_{t=0}^{\infty} \tag{1.9}$$

and prices

$$\{r_t, w_t\}_{t=0}^{\infty} \tag{1.10}$$

such that:

Households: Taking the stochastic process $\{r_t, w_t\}_{t=0}^{\infty}$ as given, the representative HH solves:

$$\max_{\{c_t, l_t\}_{t=0}^{\infty}} \quad U_0 \equiv E_0 \sum_{t=0}^{\infty} \beta^t u[C_t, 1 - l_t] \tag{1.11}$$

subject to

$$\begin{aligned} &V_{t+1} = (1+r)V_t + w_t l_t - C_t \\ &V_0 > 0 \text{ given } \lim_{t \to \infty} E_0 \Pi_{s=0}^{t} \left(\frac{1}{1+r_s} \right) V_t \geq 0 \end{aligned} \tag{1.12}$$

This problem gives the optimal $\{C_t, l_t, V_t\}_{t=0}^{\infty}$.

Firms: Taking the stochastic process $\{r_t, w_t\}_{t=0}^{\infty}$ as given, the representative firms solves:

$$\max_{\{K_t, L_t\}_{t=0}^{\infty}} \quad \pi \equiv K_t^{\alpha} (A_t L_t)^{1-\alpha} - w_t L_t - (r_t - \delta) K_t, \ \forall \text{ t} \tag{1.13}$$

This problem gives the optimal $\{K_t, L_t\}_{t=0}^{\infty}$.

$$K_t : r_t - \delta = \alpha K_t^{\alpha - 1} (A_t L_t)^{1-\alpha} \tag{1.14}$$

$$L_t : w_t = (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} \tag{1.15}$$

Markets clears for all t:

- w_t clears the labor market: $l_t = L_t$
- r_t clears the capital market: $K_t = V_t$

Optimal Allocation (the Social Planner's Problem):

$$\max_{c_t, l_t} E_0 \sum_{t=0}^{\infty} \beta^t u[C_t, 1-l_t]$$ subject to:

$$K_{t+1} = (1-\delta)K_t + I_t, \quad 0 < \delta \leq 1, \quad K_0 \geq 0$$

$$Y_t = K^{t}_{\alpha}(A_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1 \qquad Y_t = C_t + I_t + G_t$$

$$\log A_t = \log \bar{A} + gt + \tilde{A}_{t,} \quad \text{where } \tilde{A}_t = \rho \tilde{A}_{t-1} + \varepsilon_{t,} \quad \text{with } |\rho|<1 \text{ and } \varepsilon_t \sim \eta[0,1]$$

$$\log G_t = \log \bar{G} + gt + \tilde{G}_{t,} \ \textit{where } \tilde{G}_t = \rho_G \tilde{G}_{t-1} + \eta_{t,} \ \textit{with } |\rho_G|<1 \text{ and } \eta_t \sim \eta[0,1]$$

Control variables: $[c_t, L_t]$ **State variables:** $[K_t]$ **Shock:** $[A_t, G_t]$

Bellman Equation for this problem:

$$V(K_t, A_t, G_t) = \max_{C_t, l_t} \{u(C_t, 1-l_t) + \beta E_t[V(K_{t+1}, A_{t+1}, G_{t+1})]\}$$

subject to:

$$K_{t+1} = (1-\delta)K_t + K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t - G_t$$

$$\log A_t = \log \bar{A} + gt + \tilde{A}_{t,} \ \text{where } \tilde{A}_t = \rho \tilde{A}_{t-1} + \varepsilon_{t,} \ \text{with } |\rho|<1 \text{ and } \varepsilon_t \sim \eta[0,1]$$

$$\log G_t = \log \bar{G} + gt + \tilde{G}_{t,} \ \textit{where } \tilde{G}_t = \rho_G \tilde{G}_{t-1} + \eta_{t,} \ \textit{with } |\rho_G|<1 \text{ and } \eta_t \sim \eta[0,1]$$

In matrix notation:

$$\begin{bmatrix} K_{t+1} \\ \log A_t \\ \log G_t \end{bmatrix} = \begin{bmatrix} (1-\delta)K_t + K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t - G_t \\ \log \bar{A} + gt + \tilde{A}_{t,} \ \textit{where } \tilde{A}_t = \rho \tilde{A}_{t-1} + \varepsilon_t \\ \log \bar{G} + gt + \tilde{G}_{t,} \ \textit{where } \tilde{G}_t = \rho_G \tilde{G}_{t-1} + \eta_t \end{bmatrix}$$

FOC:

C$_t$: $$u_c'(C_t, 1-\ell_t) - \beta E_t\{v'(K_{t+1}, A_{t+1}, G_{t+1})\} = 0 \qquad \textbf{(1.16)}$$

or, $$u_c'(C_t, 1-\ell_t) = \beta E_t\{v'(K_{t+1}, A_{t+1}, G_{t+1})\}$$ **(IT Euler equation)**

$$l_t : \ -u_{\ell}'(C_{t,} 1-\ell_t) + \beta E_t\left\{v'(K_{t+1}, A_{t+1}, G_{t+1})[(1-\alpha)K_t^{\alpha}\ell_t^{-\alpha}A_t^{1-\alpha}]\right\} = 0 \qquad \textbf{(1.17)}$$

Use the Envelope Theorem:

$$\frac{\partial V(K_t)}{\partial K_t} = \beta E_t\left[\frac{\partial V(K_{t+1})}{\partial K_{t+1}}\frac{\partial K_{t+1}}{\partial K_t}\right] \quad \text{where} \quad K_{t+1} = (1-\delta)K_t + K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t - G_t$$

$$V'(K_t, A_t, G_t) = \beta E_t\left\{V'(K_{t+1}, A_{t+1}, G_{t+1})\left[(1-\delta) + \left\{\alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha}\right\}\right]\right\}$$

Using the FOC for C$_t$ **(1.16)** : $u_c'(C_t, 1-l_t) = \beta E_t\{v'(K_{t+1}, A_{t+1}, G_{t+1})\}$ **yields,**

$$V'(K_t, A_t, G_t) = \left[(1-\delta) + \left\{\alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha}\right\}\right]u_c'(C_t, 1-l_t) \qquad \textbf{(1.18)}$$

Iterating this expression forward to t+1:

$$V'(K_{t+1}, A_{t+1}, G_{t+1}) = \left[(1-\delta) + \left\{\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha}\right\}\right]u_c'(C_{t+1}, 1-l_{t+1}) \qquad \textbf{(1.19)}$$

Substituting **(1.18)** in **(1.16)**,

$$u_c'(C_t, 1-l_t) = \beta E_t\left\{\left[(1-\delta) + \left\{\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha}\right\}\right]u_c'(C_{t+1}, 1-l_{t+1})\right\}$$

And then replacing **(1.19)** in **(1.17)**, we have:

$$u_l'(C_{t,} 1-l_t) = \beta[(1-\alpha)K_t^{\alpha} l_t^{-\alpha} A_t^{1-\alpha}]E_t\left\{\left[(1-\delta) + \left\{\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha}\right\}\right]u_c'(C_{t+1}, 1-l_{t+1})\right\}$$

Making use of the log utility function: $u[C_t, 1-l_t] = \log C + b\ln(1-l)$ **yields:**

$$[C_t:]\ \frac{1}{C_t} = \beta E_t\left\{\left[(1-\delta) + \left\{\alpha K_{t+1}^{\alpha-1}(A_{t+1}L_{t+1})^{1-\alpha}\right\}\right]\frac{1}{C_{t+1}}\right\} \qquad (1.20)$$

$$[l_t:]\frac{b}{1-l_t} = [(1-\alpha)K_t^{\alpha} l_t^{-\alpha} A_t^{1-\alpha}]\frac{1}{C_t} \qquad (1.21)$$

- **Wages and labor supply :** From (1.21) and (1.15) we have:

$$\frac{b}{1-l_t} = [(1-\alpha)K_t^{\alpha} l_t^{-\alpha} A_t^{1-\alpha}]\frac{1}{C_t} = w_t\frac{1}{C_t} \text{ or } \frac{b}{1-l_t} = \frac{w_t}{C_t} \qquad (1.22)$$

When $\frac{w_t}{C_t} \downarrow \Rightarrow l_t \downarrow$.

- **Impacts of shocks and (booms and recessions):**
  - Transitory Technology shock: $A_t \downarrow$

$$\frac{bC_t}{1-l_t} = [(1-\alpha)K_t^{\alpha} l_t^{-\alpha} A_t^{1-\alpha}] = w_t \text{, If } A_t \downarrow \Rightarrow w_t \downarrow, C_t \downarrow l_t \downarrow \Rightarrow L^s \downarrow .$$

Labor supply and wages move in the same direction…

- Transitory Government expenditure Shock:

$$G_t \uparrow \Rightarrow l_t \uparrow \Rightarrow w_t \downarrow, L^s \uparrow$$

Labor supply and wages move in opposite directions…

- Solving the model:

$$[C_t :] \ \frac{1}{C_t} = \beta E_t \left\{ \left[(1-\delta) + \left\{ \alpha K_{t+1}^{\alpha-1} (A_{t+1} L_{t+1})^{1-\alpha} \right\} \right] \frac{1}{C_{t+1}} \right\}$$

Assuming δ=1, $C_t = (1-s)Y_t$, and $K_{t+1} = sY_t$

$$[C_t :] \ \frac{1}{(1-s)Y_t} = \beta E_t \left\{ \left[\left\{ \alpha \frac{K_{t+1}^{\alpha} (A_{t+1} L_{t+1})^{1-\alpha}}{K_{t+1}} \right\} \right] \frac{1}{(1-s)Y_{t+1}} \right\}$$

$$\Rightarrow \ \frac{1}{(1-\overline{s})Y_t} = \beta E_t \left\{ \left[\left\{ \alpha \frac{Y_{t+1}}{\overline{s}Y_t} \right\} \right] \frac{1}{(1-\overline{s})Y_{t+1}} \right\}$$

$$1 = \beta \frac{\alpha}{s}, \text{ therefore, } s = \alpha\beta$$

$$[l_t :] \frac{b}{1-l_t} = [(1-\alpha)K_t^{\alpha} l_t^{-\alpha} A_t^{1-\alpha}] \frac{1}{C_t}$$

$$[l_t :] \frac{b}{1-l_t} = (1-\alpha) \frac{Y_t}{l_t} \frac{1}{(1-s)Y_t} \quad \text{or}$$

$$[l_t :] \frac{b}{1-l_t} = (1-\alpha) \frac{1}{l_t} \frac{1}{(1-s)},$$

and replacing $\overline{s} = \alpha\beta$ in the last expression we have:

$$l_t = \frac{1-\alpha}{1-\alpha+b(1-\alpha\beta)}$$

Compare to Romer's Equation (4.37) (labor force participation rate is constant).

Log-linearization and methods:

Taking log to the equation (1.1):

$\log \mathrm{Y_t} = \alpha \log \mathrm{K_t} + (1-\alpha)[\log \mathrm{A_t} + \log \mathrm{L_t}]$, also we know

$\mathrm{K_{t+1}} = s^* Y_t$, where s* is a constant and $L_t = l^*$, therefore:

$\log \mathrm{Y_t} = \alpha \log \mathrm{s} + \alpha \log \mathrm{Y_{t-1}} + (1-\alpha)[\log \mathrm{A_t} + \log \mathrm{L_t}]$, collecting terms we have:

$$\log Y_t = [\alpha \log s^* + (1-\alpha)\log l] + \alpha \log Y_{t-1} + (1-\alpha)\log A_t \,.$$

Using (1.5) and the last expression:

$$\log Y_t = [\alpha \log s^* + (1-\alpha)\log l] + (1-\alpha)\log \bar{A} + \alpha \log Y_{t-1} + (1-\alpha)[gt + \tilde{A}_t]$$ or

$$\log Y_t - [\alpha \log s^* + (1-\alpha)\log l] - (1-\alpha)\log \bar{A} - gt = \alpha[\log Y_{t-1} - gt] + (1-\alpha)\tilde{A}_t$$

$$\tilde{Y}_t = \alpha \tilde{Y}_{t-1} + (1-\alpha)\tilde{A}_t,$$

where $\tilde{Y}_t = \log Y_t - [\alpha \log s^* + (1-\alpha)\log l] - (1-\alpha)\log \bar{A} - gt$, and $\tilde{Y}_{t-1} = \alpha[\log Y_{t-1} - gt]$

From equation (1.5) $\tilde{\mathrm{A}}_\mathrm{t} = \rho \tilde{\mathrm{A}}_{\mathrm{t}-1} + \varepsilon_{\mathrm{t},}$ with |ρ|<1 and $\varepsilon_\mathrm{t} \sim \eta[0,1]$

$\tilde{Y}_t = \alpha \tilde{Y}_{t-1} + (1-\alpha)[\rho \tilde{A}_{t-1} + \varepsilon_t\,]$ or $\tilde{Y}_t = \alpha \tilde{Y}_{t-1} + (1-\alpha)\rho \tilde{A}_{t-1} + (1-\alpha)\varepsilon_t$.

Using “lag operator” notation:

$$(1-\alpha L)\tilde{Y}_t = (1-\alpha)\tilde{A}_t$$
$$(1-\rho L)\tilde{A}_t = \varepsilon_t$$

which implies $(1-\alpha L)(1-\rho L)\tilde{Y}_t = (1-\alpha)\varepsilon_t$.