Introduction to Financial Economics

Lecture Notes 4
Ch.4-Lengwiler

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Overview of the Course

① Introduction- Finance and Economic theory- general equilibrium and macroeconomic foundations.
② Contingent Claim Economies - Commodity spaces, preferences, general equilibrium, representative agents.
③ Asset Economies- financial assets, Radner economies, Arrow-Debreu pricing, complete and incomplete markets.
④ Risky Decisions- expected utility paradigm.
⑤ Static Finance Economy- risk sharing, representative vNM agent, sdf's, equilibrium price of risk and time.
⑥ Dynamic Finance Economies- dynamic trading etc.
⑦ Taking Theory to the Data: An empirical application.

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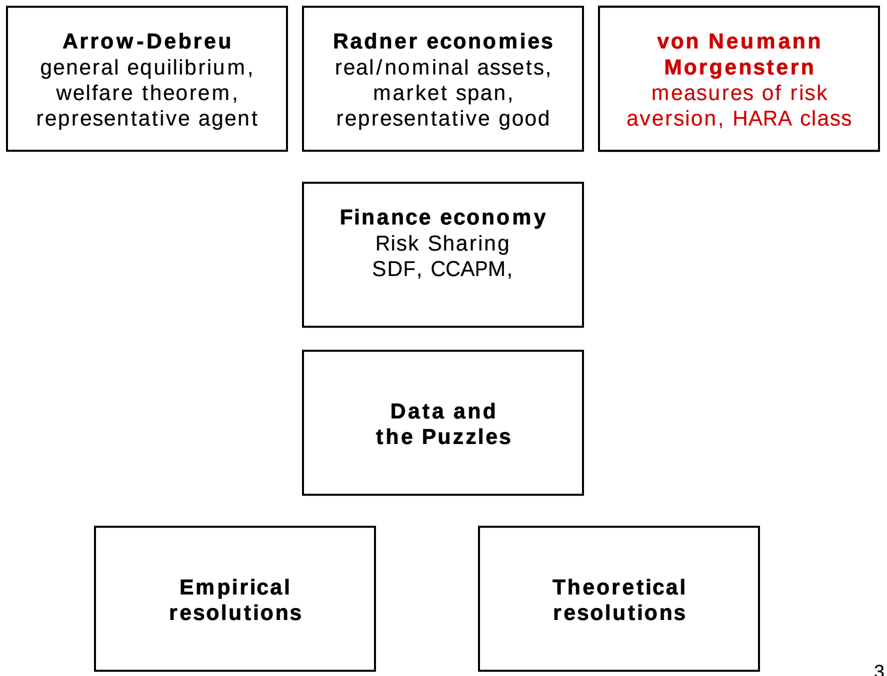

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Contingent claims + *probabilities*

- We defined commodities as being contingent on the state of the world- means that in principle we also cover decisions involving risk.
- But risk has a special, additional structure which other situations do not have: probabilities.
- We have not explicitly made use of probabilities so far.
- Theory of decision-making under risk exploits this structure to get predictions about behavior of decision-makers.
- Moved from multi-good to single good economy- but in finance we focus on risk decisions about wealth not about real assets.

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Lotteries

- Suppose you are driving to work at Renmin University from say Tainamen square !
  - If you arrive on time prize (payoff)= x , (prob.=95%).
  - If there is a traffic jam (prob=4.8%) you get nothing.
  - If you have an accident (prob =0.2%) , you get no payoff but also have to spend to repair your car.
- This lottery can be written as:[+x,0.95; 0,0.048;-y,0.02]
- Let us consider a finite set of outcomes- :[x_1,..x_s]
- The x_i 's can be consumption bundles or in our case money - the x_i 's themselves involve no uncertainty.
- We define a lottery as:

$$[x_1\pi_1;\cdots\ x_S\pi_S],\pi \geq 0,\sum_{s=1}^{S}\pi_s = 1$$

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Preferences over Lotteries

- Let us call the set of all such lotteries as $\mathcal{L}$- we now assume that agents have preferences over this set.
- So agents have a preference relation $\prec$ on $\mathcal{L}$ that satisfies the usual assumptions of ordinal utility theory; (asymmetric, negatively transitive and continuous).
- Assumptions imply that we can represent such preferences by a continuous utility function $\mathcal{V}:\mathcal{L}\rightarrow\mathbb{R}$ so that

$$\mathcal{L}\prec\mathcal{L}'\Leftrightarrow\mathcal{V}(\mathcal{L})<\mathcal{V}(\mathcal{L}')$$

- We also assume that people prefer more to less (in our case more money to less):

$$\pi_1>0,a>0\Rightarrow V([x_1\pi_1;x_2\pi_2])<V([x_1+a,\pi_1;x_2\pi_2])$$

- Let also expected value of a lottery is: $E\{L\}=\sum_{s=1}^{S}\pi_s x_s$

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What is risk aversion?

- Consider the lottery [{E(L},1}- this lottery pays E{L} with certainty or (Outcome=E(L), probability=1).
- We define attitude to risk with reference to this lottery and how agents prefer outcomes relative to this lottery.
- Risk Neutral: $\mathcal{V}$ (L)= $\mathcal{V}$ ([E{$\mathcal{L}$},1]) or the risk in the lottery L- variation in payoff between states is irrelevant to the agent- the agent cares only about the expectation of the prize.
- Risk Averse: $\mathcal{V}$ (L)< $\mathcal{V}$ ([E{$\mathcal{L}$},1]) – here the agent would rather have the average prize E{L} for sure than bear the risk in the lottery L.
- A risk averse agent is willing to give up some wealth on average in order to avoid the randomness of the prize of L.

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Certainty equivalent

- Let $\mathcal{V}$ be some utility function on $\mathcal{L}$ (set of all lotteries) and let L be some lottery with expected prize E{L}.
- The certainty equivalent of L under $\mathcal{V}$ is defined as

  $\mathcal{V}$ ([CE[L],1) = $\mathcal{V}$ ($\mathcal{L}$).
- CE(L) is the level of non-random wealth that yields the same utility as the lottery L.
- The risk premium is the difference between the expected prize of the lottery and its certainty equivalent:

  RP (L) =E{L}-CE(L).
- All of this is the same as ordinal utility theory and we have not used the additional structure in the probabilities- we now do this.

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What we are after: an *expected utility* representation

- So far we have used ordinal utility theory and we now add the idea of probabilities.
- We want to represent agent's preferences by evaluating the expected utility of a lottery.
- We need a function υ that maps the single outcome x_s to some real number $\upsilon(x_s)$, and then we compute the expected value of υ .
- Formally, function υ is the expected utility representation of $\mathcal{V}$ if :

$$V\left(\left[x_1\pi_1;\dots\ x_S\pi_S\right]\right) = \sum_{s=1}^{S} \pi_s v(x_s)$$

- Advantages υ of $\mathcal{V}$ is that it maps from $\mathbb{R}\rightarrow\mathbb{R}$ and not from $\mathcal{L}\rightarrow\mathbb{R}$ and is easier to work with mathematically.
- Von Neumann and Morgenstern first developed the use of an expected utility under some conditions- lets look at these briefly.

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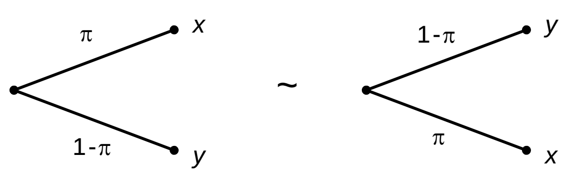
vNM axioms: state independence

- Von Neumann and Morgenstern's have presented a model that allows the use of an expected utility under some conditions.
- The first assumption is state independence.



- ➢ All that matters to an agent is the statistical distribution of outcomes.
- ➢ A state is just a label and has no particular meaning and are interchangeable (as in x and y in the diagram).

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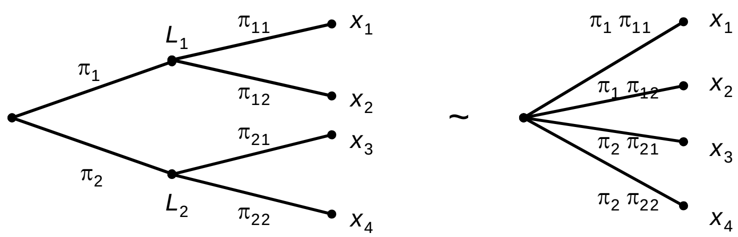
NM axioms: consequentialism

- Consider a lottery L, whose prizes are further lotteries L_1 and L_2: $L = [L_1,p_1;L_2,p_2]$- a compound lottery.
- We assume that an agent is indifferent between L and a one-shot lottery with four possible prizes and compounded probabilities.
- An agent is indifferent between the two lotteries shown in the diagram below.
- Agents are only interested in the distribution of the resulting prize, but not in the process of gambling itself.



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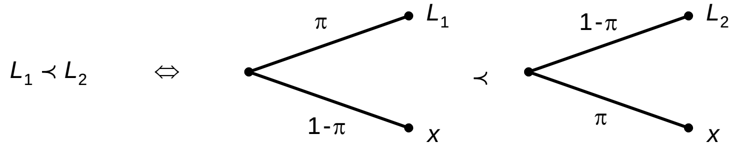
vNM axioms: irrelevance of common alternatives

- This axiom says that the ranking of two lotteries should depend only on those outcomes where they differ.
- If L_1 is better than L_2, and we compound each of these lotteries with some third common outcome x, then it should be true that $[L_1,p;x,1-p]$ is still better than $[L_2,p;x,1-p]$. The *common alternative x* should not matter.



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vNM utility theory-Some Discussion

- State-independence, consequentialism and the irrelevance of common alternatives + the assumptions on preferences $\prec$ give rise to the famous results of vNM- The utility function $\mathcal{V}$ has an expected utility representation υ such that:

$$V\left(\left[x_1\pi_1;\ldots\ x_S\pi_S\right]\right)=\sum_{s=1}^{S}\pi_s v(x_s)$$

- The utility function is on the space of lotteries $\mathcal{V}$ which represents the preference relation between lotteries and is an ordinal utility function.
- $\mathcal{V}$ (L) is an ordinal measure of satisfaction and can be compared only in the sense of ranking lotteries.
- $\mathcal{V}$ is also invariant to monotonic transformations.

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vNM utility theory-Some More Discussion

- The vNM utility function υ has more structure.
- It represents $\mathcal{V}$ as a linear function of probabilities.
- As a result υ is not invariant under an arbitrary monotonic transformation.
- It is invariant only under positive affine transformations.
- Hence vNM utility is cardinal.
- What does this mean?
- Cardinal numbers are measurements that are ordinal but whose difference can also be ordered.
- With cardinal utility we can have the following:

$$v(x_1)>v(x_2),v(x_3)>v(x_4)\Rightarrow v(x_1)-v(x_2)>v(x_3)-v(x_4)$$

- This is not necessarily the case with ordinal utility.

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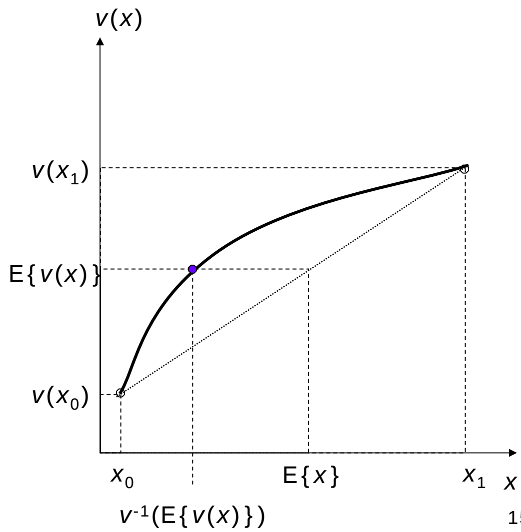
Risk-aversion and concavity-I

- The certainty equivalent is the level of wealth that gives the same utility as the lottery on average. Formally:

$$v\big(CE(x)\big) = E\{v(x)\}$$

- We can explicitly solve for the CE as:

$$CE(x) = v^{-1}\Big(E\{v(x)\}\Big)$$



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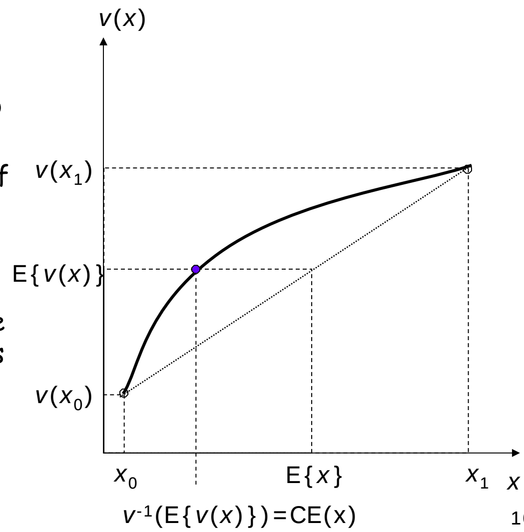
Risk-aversion and concavity-II

- We now see that the agent is risk averse iff υ is a concave function.
- Jensen's inequality: strict convex combination of two values of a function is strictly below the graph of the function then the function is concave.
- The risk premium is therefore positive and the agent is risk averse if υ is strictly concave.
- If υ'' =0, then CE(x)=E{x} and the RP=0 or risk neutrality.



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Absolute Risk Aversion

- We define the coefficient of Absolute Risk Aversion (ARA) as a local measure of the degree that an agent dislikes risk.

$$A(w) = -\frac{v''(w)}{v'(w)}$$

- A has many useful properties
  - Its is invariant under an affine transformation or if u and v are two vNM utility functions then ARA of u and v are the same.
  - We can use the ARA then for interpersonal comparisons.
  - Suppose Mr. X and Mr. Y have the same endowments but different preferences. X's utility function v is more concave than Y's (say u- is more concave) so X always demands a higher risk premium for a given level of risk.
  - Here then ARA for v(w) is larger than the ARA for u(w).

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CARA-DARA-IARA

- A utility function υ exhibits constant relative risk aversion of CARA if ARA does not depend on wealth or A(w)=0.
- υ exhibits decreasing absolute risk aversion or DARA if richer people are less absolutely risk averse than poorer ones or A'(w)<0.
- υ exhibits increasing absolute risk aversion or IARA if A'(w)>0.

What do these mean in economic terms?

- Consider a simple binary lottery- you cannot win anything but can loose $10 with 50% probability.
- CARA $\Rightarrow$ millionaire requires the same payment to enter this lottery as a beggar would.
- IARA $\Rightarrow$ millionaire requires a larger payment than the beggar!
- Millionaire takes it for a smaller payment than a beggar- most realistic case –or DARA.

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Relative Risk Aversion

- Consider another simple binary lottery- instead of losing $10 with 50% probability now we have a 50% probability of losing your wealth.
- For the beggar this amount to losing a few cents for the millionaire it may be in $100,000.
- Who requires a larger amount up front, in terms of percentage of his wealth, to enter this gamble? Not easy to answer?
- Suppose the millionaire requires $70,000- this is not unrealistic and the beggar requires 30 cents- also probable- then the millionaire requires a larger percentage of his wealth than the beggar $\Rightarrow$ millionaire is thus more relatively risk averse than the beggar.
- This is measured as Coefficient of RRA: $R(w) = w\,A(w)$
- If R is independent of wealth for CRRA utility functions.

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Precautionary Saving

- Coefficients of risk aversion measure the disutility arising from small amount of risk imposed on agents or how much an agent dislikes risk.
- Coefficients do not tell us about how the behavior of agents changes when we vary the amount of risk the agent is forced to bear.
- Example: It may be reasonable for agents to accumulate some "precautionary" saving when facing more uncertainty.
- More risk induces a more prudent agent to accumulate precautionary savings.
- Kimball's coefficient of absolute prudence: $P(w) = -\dfrac{v'''(w)}{v''(w)}$
- An agent is prudent iff this coefficient is positive.
- The precautionary motive is important because it means that agents save more when faced with more uncertainty.

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Empirical estimates

- Many studies have tried to obtain estimates of these coefficients using real-world data.
- Friend and Blume (1975): study U.S. household survey data in an attempt to recover the underlying preferences. Evidence for DARA and almost CRRA, with $R \approx 2$.
- Tenorio and Battalio (2003): TV game show in which large amounts of money are at stake. Estimate relative risk aversion between 0.6 and 1.5.
- Abdulkadri and Langenmeier (2000): farm household consumption data. They find significantly more risk aversion.
- Van Praag and Booji (2003): survey-based study done by a Dutch newspaper. They find that relative risk aversion is close to log-normally distributed, with a mean of 3.78.

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Frequently used HARA Utility functions

Utilty functions that (i) strictly increasing (ii) strictly concave (iii) DARA or $A'(w)<0$ (iv) not too large relative risk aversion ($0<R(w)<4$) for all w are the properties that are most plausible.

name	formula	*A*	*R*	*P*	*a*	*b*
affine	$\gamma_0+\gamma_1 y$	0	0	undef	undef	undef
quadratic	$\gamma_0 y - \gamma_1 y^2$	incr	incr	0	$\gamma_0/(2\gamma_1)$	−1
exponential	$-e^{-\gamma y}/\gamma$	γ	incr	γ	$1/\gamma$	0
power	$y^{1-\gamma}/(1-\gamma)$	decr	γ	decr	0	$1/\gamma$
Bernoulli	$\ln y$	decr	1	decr	0	1

- *A*, *R*, and *P* denote absolute risk aversion, relative risk aversion, and prudence. *a* and *b* will be explained later.
- All these belong to the class of HARA functions.

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The HARA class

- Most of the plausible utility functions belong to the HARA or *hyperbolic absolute risk aversion* (or *linear risk tolerance utility function*) class.
- Define absolute risk tolerance as the reciprocal of absolute risk aversion, $T := 1/A$.
- u is HARA if T is an affine function, $T(y) = a + by$.
- Merton shows that a utility function v is HARA if and only if it is an affine transformation of:

$$v(y) := \begin{cases} \ln(y+a), & \text{if } b = 1, \\ -ae^{-y/a}, & \text{if } b = 0, \\ (b-1)^{-1}(a+by)^{(b-1)/b}, & \text{otherwise.} \end{cases}$$

- DARA $\Leftrightarrow$ $b>0$; CARA $\Leftrightarrow$ $b=0$; IARA $\Leftrightarrow$ $b<0$, v is CRRA iff $a = 0$.
- Most results in finance rely on assumption of HARA utility- whether these are realistic is another matter.

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CRRA and Homotheticity

- Of the HARA class, the CRRA is a popular specification used in the asset pricing literature –both in theory and empirics.
- Why- there is some favorable empirical evidence (Friend and Blume, 1975) and it has some nice theoretical properties.
- In homothetic utility functions the marginal rates of substitution do not change along a ray through the origin- hence the composition of the optimal consumption bundle is not affected by the agent's wealth but depends on relative prices.
- RRA is independent of wealth is another property.

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Mean-Variance Utility

- Many researchers in finance (Markowitz ,Sharpe etc.) used mean variance utility functions. But is it compatible with NM theory?
- The answer is yes … approximately … under some conditions.

What are these conditions?

- υ is quadratic e.g. when $\upsilon = aw - bw^2$
- If asset returns are joint normal.
- Belongs to the linear distribution class.
- For small risks.
- We will not go into the details of these issues here.

Mean-variance: small risks

- The most relevant justification for mean-variance is probably the case of small risks.
- If we consider only small risks, we may use a second order Taylor approximation of the vNM utility function.
- A second order Taylor approximation of a concave function is a quadratic function with a negative coefficient on the quadratic term.
- In other words, any risk-averse NM utility function can locally be approximated with a quadratic function.
- But the expectation of a quadratic utility function can be evaluated with the mean and variance. Thus, to evaluate small risks, mean and variance are enough.