ECGA 7020 Macro Theory II – Midterm Answers Fordham University

Question 1. A Canonical OLG model and Dynamic Efficiency:

Suppose overlapping generations maximize, $U_t = \ln c_{1t} + \beta \ln c_{2t+1}$ and production takes place via a Cobb-Douglas production function: $F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}$ **.**

(a) Set up the Langrangian for this problem (suggestion: use one constraint for each period)

$$\mathcal{L} = \ln c_{1t} + \beta \ln c_{2t+1} + \lambda_t(w - c_{1t} - k_{t+1}) + \mu_t(r_{t+1}k_{t+1} - c_{2t+1})$$

F.O.C.: 1) $1/c_{1t} - \lambda_t = 0$
2) $\beta/c_{2t+1} - \mu_t = 0$ combining eqs. 1-3 yields the Euler equation: $c_{1t}\beta\, r_{t+1} = c_{2t+1}$
3) $-\lambda_t + \mu_t r_{t+1} = 0$

b) Solve for the workers OLG savings rate recalling that $k_{t+1} = s_t w_t$ Combining the Euler equation, $c_{1t}\beta r_{t+1} = c_{2t+1}$, with the 2nd period budget constraint, $r_{+1}k_{t+1} = c_{2t+1}$ yields:

$c_{1t}\beta\, r_{t+1} = r_{t+1}k_{t+1}$ or, $c_{1t}\beta = k_{t+1}$. The 1st period budget constraint, $w_t = c_{1t} + k_{t+1}$ plus the Euler equation implies, $w_t = (1/\beta)k_{t+1} + k_{t+1} = (1+1/\beta)k_{t+1}$ Replacing k_{t+1} in this expression with $s_t w_t$ yields

$$s_{olg} = \frac{\beta}{1+\beta} \quad or \quad \frac{1}{2+\rho}$$

c) Solve for the standard golden rule savings rate, give that c = y – I which in this model is $c = Ak^{\alpha} - k$ where k = K/L.

Find the golden rule capital stock that maximizes $c = Ak^{\alpha} - k$ by setting $\partial c/\partial k = 0$, which yields,

$k_{gr} = (\alpha A)^{\frac{1}{1-\alpha}}$ but since $w_{gr} = (1-\alpha)Ak^{\alpha}$ but from the (b) we know **$k_{gr} = s_{gr}\, w_{gr}$ or,**

$$s_{gr} = \frac{k_{gr}}{w_{gr}} = (1-\alpha)Ak_{gr}^{(\alpha-1)} \quad but\ as \quad k_{gr} = (\alpha A)^{\frac{1}{1-\alpha}} \Rightarrow \quad s_{gr} = \frac{\alpha}{1-\alpha}$$

d) Compare the OLG savings rate to the golden rule savings rate (with wages and savings all computed at k_{gr}). What conditions on preferences and technology make dynamic inefficiency more likely in this model?

Dynamic inefficiency occurs when $s_{olg} > s_{gr}$. This is more likely to occur when β is high and α is low. For example, if $\beta = .9$ (or $\rho = .1$) and $\alpha < .32$ the OLG savings rate is dynamically inefficient

Question I-3. Dynamic Programming CA Policy Functions: Assuming log utility,

$u(c_t) = \ln c_t$, $$Max\sum_{t=0}^{\infty}\beta u(c_t) \quad (0.1)$$

$W_{t+1} = (1+r)(W_t - c_t)$ assuming r_t is the fixed world interest rate and wealth at date t is,

$W_t = (1+r)B_t + \tilde{Y}_t$ *where* $\tilde{Y}_t = \sum_{S=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s$ *is the present value of future* output at date t.

(a) Uses the form of the Bellman equation shown below and solve this problem using W_{t+1} as the control variable:

$$V(W_t) = \max_{c\in(0,W)} u(c) + \beta V(W_{t+1})$$ or,

Be sure to show each step in the derivation. Recall the basic strategy of Adda and Cooper or Obstfeld and Rogoff, rewriting $u(c_t)$ as a function W_{t+1}. Note the Euler equation for this problem. You can check your answer however, you want, but only write out the forward looking solution using W_{t+1} as your control variable. You want to end up with a stationary policy function of the form: $W_{t+1} = \varphi(W_t)$.

Rewriting $c_t = W_t - W_{t+1}/(1+r)$ $$V(W_t) = \max_{W_{t+1}\in(0,W)} u\left[W_t - \frac{W_{t+1}}{(1+r)}\right] + \beta V(W_{t+1})$$

The FOC for this problem is $$\frac{-u'(c_t)}{1+r} + \beta V'(W_{t+1}) \quad \text{or} \quad V'(W_{t+1}) = \frac{1}{\beta c_t (1+r)}$$

Conjecture: $W_{t+1} = F \ln(W_t)$ so that $V'(W_t) = F/W_t$ Which also implies,

$$u'(c_t) = \beta(1+r)V'(W_{t+1}) \quad \text{or that} \quad \frac{1}{c_t} = \frac{\beta(1+r)F}{W_{t+1}} \Rightarrow \quad W_{t+1} = \beta(1+r)Fc_t$$

However, we also know $W_{t+1} = (1+r)(W_t - c_t) \Rightarrow \beta(1+r)Fc_t = (1+r)(W_t - c_t)$ s that, $c_t = \frac{W_t}{1+\beta F}$

Substituting this conjecture back into the Bellman equation,

$V(W_t) = u(c_t) + \beta V(W_{t+1})$ implies, $$E + F\ln(W_t) = \ln(c_t) + \beta[E + F\ln(\beta(1+r)Fc_t) \ \text{where} \ c_t = \frac{W_t}{1+\beta F}$$

$$E + F\ln(W_t) = (1+\beta F)\ln(W_t) + \beta\{E + F\ln(\frac{\beta(1+r)F}{1+\beta F})\} - \ln(1+\beta F)$$, then collecting terms we have:

$F=1/(1-\beta)$, and $E=1/(1-\beta)[\beta/(1-\beta)\ln[\beta(1+r)]+\ln(1-\beta)]$

Replacing F in c_t we have: $c_t = (1-\beta)W_t$ and $W_{t+1} = \beta(1+r)W_t$

b) Use $W_{t+1} – W_t$ to derive an expression for the current account, $CA_{t+1} = B_{t+1} – B_t$.

$W_{t+1} – W_t = [(1+r)\beta-1]$ Wt.

From ($W_t = (1+r)B_t + \tilde{Y}_t \quad where \quad \tilde{Y}_t = \sum_{S=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s \quad is\ the\ present\ value\ of\ future$) we have that

$W_{t+1} – W_t = (1+r)(B_{t+1} – B_t)-Y_t$, Therefore:

$[(1+r)\beta-1]$ Wt = $(1+r)(CA_t)-Y_t$ and finally $CA_t = [\beta-1/(1+r)]$ Wt + $Y_t / (1+r)$

c) Discuss how a fall in the interest rate r affects optimal growth of wealth, consumption and the current account. Compare this result with the results obtained in class for CRRA utility:

$$c_t = \left[1-\beta^{\sigma}(1+r)^{\sigma-1}\right]W_t \qquad and \qquad W_{t+1} = \left[\beta^{\sigma}(1+r)^{\sigma}\right]W_t$$

where $\sigma = 1/\theta$ in Romer's notation (the IES). Intuitively, why does the fall in r have a different effect depending on the utility function?

* Log utility

Optimal growth of wealth: $\frac{W_{t+1}}{W_t} = \beta(1+r)$, If r decreases then optimal growth decreases.

Consumption: $c_t = (1-\beta)$Wt does not depends on r.

Current Account: $CA_t = [\beta-1/(1+r)]$ Wt + $Y_t / (1+r)$ where $\frac{\partial(CA_t)}{\partial r} = \frac{1}{(1+r)^2}(Wt - Y_t)$ therefore if r decreases CA increases.

** CRRA utility

Optimal growth of wealth: $\frac{W_{t+1}}{W_t} = \beta^{\sigma}(1+r)^{\sigma}$, $\frac{\partial \frac{W_{t+1}}{W_t}}{\partial r} = \sigma\beta^{\sigma}(1+r)^{\sigma-1}$ If r decreases then optimal growth decreases (when $\sigma > 1$) and If r decreases then optimal growth increases (when $\sigma < 1$).

Consumption: $c_t = \left[1-\beta^{\sigma}(1+r)^{\sigma-1}\right]W_t$, $\frac{\partial c_t}{\partial r} = \left[-\beta^{\sigma}(\sigma-1)(1+r)^{\sigma-2}\right]W_t$. If r decreases consumption increases (σ>2). If r decreases consumption decreases (σ<2).

Current Account: $CA_t = (\beta^{\sigma}(1+r)^{\sigma-1} - \frac{1}{1+r})W_t + \frac{Y_t}{1+r}$ where

$$\frac{\partial(CA_t)}{\partial r} = [\beta^{\sigma}(\sigma-1)(1+r)^{\sigma-2}]W_t + [\frac{1}{(1+r)^2}](W_t - Y_t).$$

(d) Use Bellman's principle of optimality to explain why choosing maximizing with respect to W_{t+1} automatically maximizes today's consumption c_t and all future consumption. How does this

way of thinking about optimization generate a stationary policy function (if one exists)? The answer to this question will prove for adding optimizing under uncertainty.