ECGA 7020: Consumption SDP example: Further results

Part 2 Consumption Euler Equations and Asset pricing

(see B&F Chapter 6, 10 and Romer 7.5):

Consumption when returns are risky: $Max \sum_{t=0}^{\infty} \beta^t u(C_t)$

Setting up the Bellman equation, $V(W_t) = \max_{c_t h_t} \{u(C_t) + \beta E_t[V(W_{t+1})]\}$

Where $W_{t+1} = R_t(W_t + Y_t - C_t)$ and $R_t = [(1+r_t)h_t + (1+z_t)(1-h_t)]$ The total return on investment is r_t the risk free rate of return times $h_{t.}$ the portfolio share of risk free assets plus z_t times $(1- h_{t.})$ where z_t is the random rate of return on the risky asset, unknown at date t.

2. With quadratic utility consumption follows a random walk

From the previous handout $\mathbf{E_t\ [(u'_{t+1})\ (1+ r_t)] = E_t\ [(u'_{t+1})(1+ z_t)]}$

and Keynes-Ramsey or intertemporal Euler equation is,

$$u'(C_t)=\beta(1+ r_t)E_t\ [u'(C_{t+1})\ ], \qquad \textbf{(2.1) and,}$$

$$u'(C_t) = \beta E_t[\ (1+z_t)u'(C_{t+1})]. \qquad \textbf{(2.2)}$$

Assuming quadratic utility $u(C_t) = aC_t - bC_t^2$ and that $1+r_t = \frac{1}{\beta}$, (2.1) becomes

$u'(C_t)=\beta(1+ r_t)u'(C_{t+1})+\bar{e}_{t+1}$, where $E_t(\bar{e}_{t+1} \mid t) = 0$, yields

$C_{t+1} = C_t + e_{t+1}$, where $e_{t+1} = \left[\frac{1}{2b}\right]\bar{e}_{t+1}$ and $E_t(e_{t+1} \mid t) = 0$.

Consumption follows a random walk: given c_t no other variable known at time t should help predict consumption in c_{t+1}.

Deriving the consumption CAPM Model (see B&F chapter 10 page 507):

From $\mathbf{E_t\ [(u'(C_{t+1}))(r_t - z_t)] = 0}$ **and** $\mathbf{E_t\ [(u'(C_{t+1}))(z_t - r_t)] = 0}$

$$\mathbf{0 = E_t[u'(C_{t+1})]E_t(z_t - r_t) + cov[u'(C_{t+1})z_t]}$$

$$\mathbf{0 = E_t[u'(C_{t+1})](E_t(z_t) - r_t) + cov[u'(C_{t+1})z_t]} \quad \text{or}$$

$$\mathbf{E_t(z_t) = r_t - \frac{cov[u'(C_{t+1})z_t]}{E_t[u'(C_{t+1})]}}. \tag{2.3}$$

From CAPM to Finance Betas and Asset pricing (B&F page 508)

Suppose there exists an asset m whose returns are perfectly negative correlated with consumption. Holding such an asset would prove a perfect hedge for smoothing uncertainty, so it becomes the hypothetical benchmark against which all other assets are compared. That is if, $u'(C_{t+1}) = -\gamma z_{mt}$, then $\mathbf{cov[-\gamma z_{mt} z_t] = -\gamma\, cov(z_{mt} z_t)}$.

Adding the asset m to the CAPM equation above yields:

$\mathbf{E_t(z_{mt}) = r_t - \frac{cov[u'(C_{t+1})z_{mt}]}{E_t[u'(C_{t+1})]}}$, the last but since $u'(C_{t+1})\ and\ z_{mt}$ are perfectly negatively correlated, then

$$\mathbf{E_t(z_{mt}) = r_t + \frac{\gamma\, var(z_{mt})}{E_t[u'(C_{t+1})]}} \quad \textbf{or} \quad \mathbf{E_t[u'(C_{t+1})] = \frac{\gamma\, var(z_{mt})}{E_t(z_{mt}) - r_t}},$$

Finally substituting and replacing this expression back into the CAPM equation (2.3) we get,

$$\mathbf{E_t(z_t) = r_t - \frac{cov[u'(C_{t+1})z_t]}{\gamma\, var(z_{mt})}\left[E_t(z_{mt}) - r_t\right]}, \text{ also}$$

finally if we replace $u'(C_{t+1}) = -\gamma z_{mt}$ (quadratic utility) in the equation we have:

$$\mathbf{E_t(z_t) = r_t + \frac{cov[z_{mt} z_t]}{var(z_{mt})}\left[E_t(z_{mt}) - r_t\right]}.$$

Precautionary saving (see Blanchard and Fischer Chapter 6 page 288)

Assume r_t is constant and $1+r_t = \frac{1}{\beta}$, therefore, from (2.1) $E_t\ [u'(c_{t+1})\ ] = u'(c_t)$

consumer maximizes $\quad \max E\left[\sum_{t=0}^{T-1}\left(-\frac{1}{\alpha}\right)e^{(-\alpha C_t)}\mid t=0\right]$ (2.4)

subject to $W_{t+1} = W_t + Y_t - C_t$

and

$Y_t = Y_{t-1} + e_t$, where $e_t \sim N(0,\sigma^2)$.

Solution $u'(C_t) = E(u'(C_{t+1}))$ since $1+r_t = \frac{1}{\beta}$

$$E_t\ [e^{(-\alpha C_{t+1})}] = e^{(-\alpha C_t)} \qquad (2.5)$$

Since C_t is log normally distributed with mean $E(C_t) = e^{\mu+\sigma/2}$ where $\sigma^2_{C_t}$ we have:

$$\exp\left(E(-\alpha C_{t+1}) + \frac{\alpha^2\sigma^2{}_{C_{t+1}}}{2}\right) = \exp(-\alpha C_t) \text{ or } E(-\alpha C_{t+1}) + \frac{\alpha^2\sigma^2{}_{C_{t+1}}}{2} = -\alpha C_t\text{, therefore}$$

$$E_t(C_{t+1}) = C_t + \frac{\alpha\sigma^2{}_{C_{t+1}}}{2}$$

$$E_t(C_{t+1}) = E_t(C_t) + \frac{\alpha\sigma^2{}_{C_{t+1}}}{2} + E_t e_t\text{,}$$

Finally we obtain the equation from page 290 of B&F chapter 6:

$$C_{t+1} = C_t + \frac{\alpha\sigma^2{}_{C_{t+1}}}{2} + e_t\,\text{.}$$

$$C_t = \left(\frac{1}{T-t}\right)W_t + Y_t - \frac{\alpha(T-t-1)\sigma^2}{4}\,. \qquad \textbf{(2.6)}$$

Where the savings rate increases with the variance of consumption of up until date T with all precautionary savings (the third term) in consumed (note that as t → T a larger share of wealth is consumed and the third term suddenly becomes negative when t = T because there is no future to be cautious about). Hence C_t jumps at T as precautionary savings (the third term in 2.6) and all remaining wealth is consumed.

II. Lucas Asset-Pricing Model (see B&F Chapter 10 page 510-511):

Identical infinitely lived consumers who want to maximize $E\left[\sum_{t=0}^{\infty} \beta^t u(C_t)\right]$.

In any period the consumer receives dividends on the assets that he holds so that he decides how much to consume and what assets to hold into the next period. Let D_t the dividends that the consumer receives for this investment. Let S_{it} be the quantity of asset i that the consumer holds between t and t+1. In matrix notation S_t is the nx1 vector of S_{it}.

The budget constraint can then be written as:

$C_t + p_t S_{t+1} = p_t S_t + d_t S_t$.

If we write the last expression $p_t S_{t+1} = p_t S_t + d_t S_t - C_t$, and multiply the LHS and RHS by ($p_{t+1} + D_{t+1}$) we have: $(p_{t+1} + d_{t+1}) p_t S_{t+1} = (p_{t+1} + d_{t+1})[(p_t + d_t)S_t - C_t]$ or

$(p_{t+1} + D_{t+1})S_{t+1} = \left(\frac{p_{t+1} + D_{t+1}}{p_t}\right)[(p_t + d_t)x_t - C_t]$ where $R_t = \left(\frac{p_{t+1} + d_{t+1}}{p_t}\right)$, and $W_t = (p_t + d_t)x_t$.

therefore: $W_{t+1} = R_t[W_t - C_t]$, but if we consider the income labor Y_t, we have $W_{t+1} = R_t[W_t + Y_t - C_t]$. In this context, the problem of consumers is given by:

$$E\left[\sum_{t=0}^{\infty} \beta^t u(C_t)\right]$$

subject to $W_{t+1} = R_t[W_t - C_t]$

The solution is given by: $u'(C_t) = \beta E_t\left[\frac{p_{t+1} + d_{t+1}}{p_t}\right] u'(C_{t+1})$ or

$$p_t = \beta E_t\left[p_{t+1} + d_{t+1}\right]\frac{u'(C_{t+1})}{u'(C_t)}.$$

In equilibrium $S_{t+1} = S_t$, and from $C_t + p_t S_{t+1} = p_t S_t + d_t S_t$, we have $C_t = d_t S_t$ under the assumption that $S_t = 1$, it yields $C_t = d_t$. Therefore

$$p_t = \beta E_t\left[p_{t+1} + d_{t+1}\right]\frac{u'(d_{t+1})}{u'(d_t)}$$

For example with CRRA: $u(C_t)=\frac{C_t^{1-\theta}}{1-\theta}$

$$p_t=\beta E_t\left[p_{t+1}+d_{t+1}\right]\frac{d_{t+1}^{-\theta}}{d_t^{-\theta}} \text{ or } p_t=\beta E_t\left[p_{t+1}+d_{t+1}\right]\frac{d_{t+1}^{-\theta}}{d_t^{-\theta}}=\beta E_t\left[p_{t+1}\frac{d_{t+1}^{-\theta}}{d_t^{-\theta}}\right]+E_t\left[\frac{d_{t+1}^{1-\theta}}{d_t^{-\theta}}\right].$$

Solving by successive forward iteration and gathering terms yields:

$$p_t=[\phi d_1^{1-\theta}+(1-\phi)d_0^{1-\theta}]\left(\frac{\beta}{1-\beta}\right)d_t^{\theta}.$$