ECGA 7020: Stochastic Dynamic Programming Example (see B&F Chapter 6):

Consumption when returns are risky: $Max \sum_{t=0}^{\infty} \beta u(c_t)$

Setting up the Bellman equation, $V(W_t) = \max_{c_t h_t} \{u(c_t) + \beta E_t[V(W_{t+1})]\}$ (1.1)

Where $W_{t+1} = R_t(W_t + Y_t - c_t)$ and $R_t = [(1+r_t)h_t + (1+z_t)(1-h_t)]$ The total return on investment

is r_t the risk free rate of return times h_t the portfolio share of risk free assets plus z_t times $(1- h_t)$ where z_t is the random rate of return on the risky asset, unknown at date t.

Taking the F.O.C. with respect w/r to the control variable c_t (see B&F page 281) yields,

$$\frac{\partial(u(c_t))}{\partial c_t} + \beta E_t\left(\frac{\partial V(W_{t+1})}{\partial W_{t+1}}\frac{\partial(W_{t+1})}{\partial c_t}\right) = 0 \quad \text{or} \quad \frac{\partial(u(c_t))}{\partial c_t} - \beta E_t\left(\frac{\partial V(W_{t+1})}{\partial W_{t+1}}R_t\right) = 0$$

$$or\ u'(c_t) = \beta E_t[R_t V'(W_{t+1})] \qquad (1.2)$$

where (1.2) is a stochastic version of the usual Euler equation. The FOC for the asset share h_t is

$$\beta(\mathrm{W_t + Y_t - c_t})E_t\left(\frac{\partial V(W_{t+1})}{\partial W_{t+1}}(r_t - z_t)\right) = 0 \quad \text{or} \quad E_t\left(\frac{\partial V(W_{t+1})}{\partial W_{t+1}}(r_t - z_t)\right) = 0$$

$$E_t [V'(W_{t+1})(r_t - z_t)] = 0 \qquad (1.3)$$

Using the Envelope Theorem: Given a specific functional form for $u(c_t)$ we could solve for the policy function explicitly. However, we can go a bit further using some conditions in the neighborhood of the optimum in the form of an envelope theorem or what at Romer likes to the "informal perturbation" method. Intuitively, what would one more tiny increment of wealth mean for consumer welfare. One way to answer this question is just to perturb wealth a bit on both sides of (1.1) to see what happens. In other words, adding a little wealth added to the value function at or the near the optimum c change $u(c_t)$ by the $u'(c^*)$ where c^* is the optimal c_t.

$$\frac{\partial V(W_t)}{\partial W_t} = \beta E_t[\frac{\partial V(W_{t+1})}{\partial W_{t+1}}\frac{\partial W_{t+1}}{\partial W_t}] \quad \text{or} \quad \frac{\partial V(W_t)}{\partial W_t} = \beta E_t[\frac{\partial V(W_{t+1})}{\partial W_{t+1}}R_t]$$

Finally from (1.2) $\frac{\partial V(W_t)}{\partial W_t} = \beta E_t[\frac{\partial V(W_{t+1})}{\partial W_{t+1}}R_t] = \frac{\partial(u(c_t))}{\partial c_t}$ that is, $V'(W_t) = u'(c_t)$.

Using $V'(W_t) = u'(c_t)$ and $V'(W_{t+1}) = u'(c_{t+1})$ we can rewrite (1.2) and (1.3) as

$$u'(c_t) = \beta E_t[R_t V'(W_{t+1})] = \beta E_t[R_t u'(c_{t+1})] = E_t[\beta R_t u'(c_{t+1})] \quad \Rightarrow$$

$$u'(c_t) = E_t[\ \beta\ R_t\ u'(c_{t+1})]\ . \qquad (1.4)$$

Returning to the F.O.C. for h_t, $\mathbf{E_t\ [V'(W_{t+1})(\ r_t - z_t)] = 0} \quad \Rightarrow \mathbf{E_t\ [(u'_{t+1})[(1+ r_t)-(1+ z_t)] = 0} \Rightarrow$

$E_t\,[u'(c_{t+1})*(1+r_t)] = E_t\,[u'(c_{t+1})*(1+z_t)]$. (1.5)

recalling that $R_t = [(1+r_t)h_t + (1+z_t)(1-h_t)]$ we can substitute 1.5 in 1.4 to obtain 1.5'

$$u'(c_t) = \beta E_t[R_t * V'(W_{t+1})] = \beta E_t[R_t * u'(c_{t+1})]$$
$$u'(c_t) = \beta E_t\left\{[(1+r_t)*h_t + (1+z_t)*(1-h_t)]*u'(c_{t+1})\right\}$$
$$u'(c_t) = \beta E_t[(1+r_t)h_t * u'(c_{t+1})] - \beta E_t[(1+z_t)h_t * u'(c_{t+1})] + \beta E_t[\ (1+z_t)*u'(c_{t+1})]$$

Finally from (1.5) we have: $u'(c_t) = \beta E_t[\ (1+z_t)*u'(c_{t+1})]$ and $u'(c_t) = \beta(1+r_t)*E_t\,[u'(c_{t+1})]$

which is same as B&F report in chapter 6 p. 282 eq 5' and 5''.

To go further, we have three options: (1) Insert an explicit quadratic utility function in 1.5' to derive the consumption CAPM for example (see Blanchard Chapter 10 or Romer 7.5). (2) Insert an explicit distribution for z_t and a CRRA utility function for example, and see how an increase in uncertainty affects consumption decisions (see Dixit (2000) chapter 10 and the handout based on that solution). (3) Solve this asset shares equation using log utility and insert it into a dynamic current account model as in Basu and McLeod(2000) or Obstfeld and Rogoff Chapter 5.

Derivation of Drazen's Envelope Theorem: Making use of the F.O.C. for W_{t+1} Drazen (2000) suggests the following envelope theorem which involves no t+1 or E_t terms (see p.34 equation 2.14 – and see below for a derivation of this version of the envelope theorem).

$$\frac{\partial u(f(k_t)-k_{t+1})}{\partial u}\frac{\partial(f(k_t)-k_{t+1})}{\partial k_{t+1}} + \beta\frac{\partial V(k_{t+1})}{\partial k_{t+1}} = 0 \quad \text{or} \quad u'((f(k_t)-k_{t+1})) = \beta\frac{\partial V(k_{t+1})}{\partial k_{t+1}}.$$

$$\frac{\partial V(W_t)}{\partial W_t} = \frac{\partial u(f(W_t)-I_t)}{\partial u}\frac{\partial(f(W_t)-I_t)}{\partial W_t} + \beta E_t\left(\frac{\partial V(W_{t+1})}{\partial W_{t+1}}\frac{\partial W_{t+1}}{\partial W_t}\right) \quad \text{or}$$

$V'(W_t) = u'[f(W_t) - (1/R_t)W_{t+1}]\,f'(W_t)$ where in our case $f(W_t) = W_t + Y_t$ and $c_t = f(W_t) - (1/R_t)W_{t+1}$ which implies simply that $V'(W_t) = u'(c_t)$ since in our case $F'(W_t) = 1$ (but this could well be $f(k_t)$).

Here is a brief derivation of Drazen's version of the envelope theorem:

$$\frac{\partial V(k_t)}{\partial k_t} = \frac{\partial u(f(k_t)-k_{t+1})}{\partial u}\frac{\partial(f(k_t)-k_{t+1})}{\partial k_t} + \beta\frac{\partial V(k_{t+1})}{\partial k_{t+1}}\frac{\partial k_{t+1}}{\partial k_t} \quad \text{or}$$

$$\frac{\partial V(k_t)}{\partial k_t} = \frac{\partial u(f(k_t)-k_{t+1})}{\partial u}\frac{\partial(f(k_t))}{\partial k_t} - \frac{\partial u(f(k_t)-k_{t+1})}{\partial u}\frac{\partial(k_{t+1})}{\partial k_t} + \beta\frac{\partial V(k_{t+1})}{\partial k_{t+1}}\frac{\partial k_{t+1}}{\partial k_t} \quad \text{collecting terms:}$$

$$\frac{\partial V(k_t)}{\partial k_t} = \frac{\partial u(f(k_t)-k_{t+1})}{\partial u}\frac{\partial(f(k_t))}{\partial k_t} + \frac{\partial k_{t+1}}{\partial k_t}\left(\beta\frac{\partial V(k_{t+1})}{\partial k_{t+1}} - \frac{\partial u(f(k_t)-k_{t+1})}{\partial u}\right).$$

The expression in brackets 0 because of it is the FOC from **Error! Reference source not found.**: