Final Exam ECGA7020 **D. McLeod Answer Key** **Fall 2005**

1. Simple RBC model (16 points)
2. Precautionary savings. (16 points)
3. Choosing a Central Banker (16 points)
4. **Portfolio choice (12 points)**
5. **Asset pricing (12 points)**
6. *Ramsey model with taxes (6 points)*

5. **Lucas Asset Pricing:** The economy consists of identical infinitely-lived consumers, the representative consumer maximizes

$E_t\left[\sum_{t=0}^{\infty}\beta^t u(C_t)\right]$. Subject to $W_{t+1} = R_t[W_t + Y_t - C_t]$, where R_t is given by $R_t = \frac{p_{t+1}+d_{t+1}}{p_t}$. Let p_t to be the current price of an asset, and d_{t+1} the dividends received by holding the asset from t to t+1 so that the return of this investment is given R_t. Let W_t the current financial wealth of the consumer.

a) Using Dynamic programming, set up the Bellman Equation

b) Derive the first order condition and the envelope theorem.

c) Derive the Basic Pricing Equation $p_t = E_t\left[\beta\left(\frac{u'(C_{t+1})}{u'(C_t)}\right)[p_{t+1}+d_{t+1}]\right]$, and assuming that $C_t = d_t$ (Competitive equilibrium and considering only one unit of asset), state the Lucas Asset Pricing Equation.

d) Solving the Basic Pricing Equation by forward iteration yields $p_t = E_t\left[\beta\left(\frac{u'(C_{t+1})}{u'(C_t)}\right)d_{t+1}\right]$

Assuming log utility, $u(C_t) = \log(C_t)$, derive the asset price p_t.

Solution a) The Bellman equation for this question is,

$$V_t(W_t) = u(C_t) + \beta E_t[V_{t+1}(W_{t+1})] \quad (0.1)$$

b) The first order condition w.r. to consumption is,

$$[C_t :]\; u'(C_t) + \beta E_t\left[V'_{t+1}(W_{t+1})\frac{\partial W_{t+1}}{\partial C_t}\right] = 0 \quad (0.2)$$

But since $R_t = \frac{p_{t+1}+d_{t+1}}{p_t}$

$$u'(C_t) = \beta E_t\left[V'_{t+1}(W_{t+1})\left(\frac{p_{t+1}+d_{t+1}}{p_t}\right)\right] \quad (0.3)$$

Now recall the basic envelope theorem:

$$V'_t(W_t) = u'(C_t)\frac{\partial C_t}{\partial W_t} + \beta E_t\left[V'_{t+1}(W_{t+1})\frac{\partial W_{t+1}}{\partial W_t}\right] + \beta E_t\left[V'_{t+1}(W_{t+1})\frac{\partial W_{t+1}}{\partial C_t}\frac{\partial C_t}{\partial W_t}\right] \quad (0.4)$$

From (0.2) we have

$$V_t^{'}(W_t) = \beta E_t\left[V_{t+1}^{'}(W_{t+1})\frac{\partial W_{t+1}}{\partial W_t}\right], \tag{0.5}$$

and using (0.3) to replace the r.h.s. of (1.19) yields:

$$V_t^{'}(W_t) = u'(C_t) \tag{0.6}$$

a) Using (0.3) and (0.6) we have

$$u'(C_t) = \beta E_t\left[u^{'}(C_{t+1})\left(\frac{p_{t+1}+d_{t+1}}{p_t}\right)\right], \tag{0.7}$$

Finally we have the basic asset pricing equation:

$$p_t = E_t\left[\beta\left(\frac{u^{'}(C_{t+1})}{u'(C_t)}\right)(p_{t+1}+d_{t+1})\right] \tag{0.8}$$

And when we assume: $C_t = d_t$ as we get the famous Lucas Asset Pricing Equation:

$$p_t = E_t\left[\beta\left(\frac{u^{'}(d_{t+1})}{u'(d_t)}\right)(p_{t+1}+d_{t+1})\right] \tag{0.9}$$

d) Given the log utility function $u(C_t) = \log(C_t)$, then $u'(C_t) = \frac{1}{C_t}$ so that

$$p_t = E_t\left[\sum_{j=1}^{\infty}\beta^j\left(\frac{d_t}{d_{t+j}}\right)d_{t+j}\right] = \left(\frac{\beta}{1-\beta}\right)d_t \tag{0.10}$$

The price of the stock depends only in the current dividend, not on expected future dividends.

Explain the intuition and novelty of this asset pricing equation. Asset prices depend on the present value of expected future income or dividends, increases in future dividends have two effects on today's consumption, first higher expected income increases today's wealth and consumption. Second, higher dividends increase the rewards to saving (investing in stocks). With log utility these two effects exactly offset one another, leading to no change in current consumption, hence only current dividends matter for asset prices (the value of that stock to an individual household today). Is the log utility function plausible given the empirical evidence on consumption and savings? Log utility is plausible to the extent to which savings and labor force participation change slowly over time. It is also true that savings are relatively insensitive to interest rate changes, which is again consistent with log utility. In what sense is this asset pricing formula analogous to a certainty equivalent price, and to q_t with quadratic adjustment costs? Since future income does not enter into the asset pricing rule, variability in future consumption does not matter as well, so consumption under uncertainty is identical to that under certainty. What makes this asset pricing model economics and not finance? The price of the asset derives from utility maximization over time, and depends on the marginal utility of consumption for each household.

4. Portfolio Choice: (this question is taken directly from Adda and Cooper Chapter 6 page 145). Newly minted PhD chooses to save in one or both of two assets, one risky and one safe. With no initial wealth, all savings must be financed out of current income and can save current income through these two assets. The safe risk free non-stochastic gross return is R^S. The risky asset $\tilde{R}^r$ and a mean return of $\bar{R}^r$. Let a^r and a^s are holdings of asset type j = r, s since there is only the present and future. Assets' prices are normalized to 1 in period 1.

The portfolio choice problem can then be written as

$$\max_{a^r, a^s} u(y_1 - a^r - a^s) + \beta E_1\left[u(\tilde{R}^r a^r + R^s a^s + y_2)\right]$$

a) Assuming that y_l is known with certainty. Derive the first-order conditions for c_0

$$\max_{a^r, a^s} u(y_0 - a^r - a^s) + \beta E_0\left[u(\tilde{R}^r a^r + R^s a^s + y_1)\right]$$

FOC: $\left[a^r :\right] \; u'(y_0 - a^r - a^s)(-1) + \beta E_0\left[u'(\tilde{R}^r a^r + R^s a^s + y_1)\tilde{R}^r\right] = 0$ **(0.11)**

$\left[a^s :\right] \; u'(y_0 - a^r - a^s)(-1) + \beta E_0\left[u(\tilde{R}^r a^r + R^s a^s + y_1)R^s\right] = 0$ (0.12)

In Equilibrium we would expect that households to be indifferent to holding an extra unit of either asset, so we can equate first-order conditions (0.11) and (0.12) such that:

$$\beta E_0\left[u(\tilde{R}^r a^r + R^s a^s + y_1)R^s\right] = \beta E_0\left[u'(\tilde{R}^r a^r + R^s a^s + y_1)\tilde{R}^r\right]$$

$$R^s E_0\left[u(\tilde{R}^r a^r + R^s a^s + y_1)\right] = E_0\left[u'(\tilde{R}^r a^r + R^s a^s + y_1)\tilde{R}^r\right]$$

$$R^s E_0\left[u(\tilde{R}^r a^r + R^s a^s + y_1)\right] = E_0\left[u'(\tilde{R}^r a^r + R^s a^s + y_1)\right]E_0\left[\tilde{R}^r\right] + \operatorname{cov}\left[u'(\tilde{R}^r a^r + R^s a^s + y_1), \tilde{R}^r\right]$$

$$R^s = \frac{E_0\left[u'(\tilde{R}^r a^r + R^s a^s + y_1)\right]E_0\left[\tilde{R}^r\right]}{E_0\left[u(\tilde{R}^r a^r + R^s a^s + y_1)\right]} + \frac{\operatorname{cov}\left[u'(\tilde{R}^r a^r + R^s a^s + y_1), \tilde{R}^r\right]}{E_0\left[u(\tilde{R}^r a^r + R^s a^s + y_1)\right]}$$

$$R^s = E_0\left[\tilde{R}^r\right] + \frac{\operatorname{cov}\left[u'(\tilde{R}^r a^r + R^s a^s + y_1), \tilde{R}^r\right]}{E_0\left[u(\tilde{R}^r a^r + R^s a^s + y_1)\right]}$$

$$R^s = \bar{R}^r + \frac{\operatorname{cov}\left[u'(\tilde{R}^r a^r + R^s a^s + y_1), \tilde{R}^r\right]}{E_0\left[u(\tilde{R}^r a^r + R^s a^s + y_1)\right]} \qquad \textbf{(1.38)}$$

b) Under which conditions the consumer wants to hold both types of assets, or only one asset,
Since we expect households to be compensated for taking on additional risk, the risky asset will be held on if the covariance term in the numerator of (1.38) is negative, in other words only if the expected return on the safe asset is lower than that of the risky assets. Risk is only worth taking to the extent that it hedges expected fluctuation in income or consumption (as indicated by the negative covariance term).

c) What are the implications if the average return on both assets is equal?

Households will only hold the safe asset.

d) What is the arbitrage condition between risky asset and non-stochastic asset?

The spread between the two asset returns will be bid down until it is exactly equal to the covariance divided by the expected utility of income in the next period.