Final Exam ECGA7020 — D. McLeod Answer Key — Fall 2005

1. **Simple RBC model (16 points)**
2. **Precautionary savings. (16 points)**
3. Choosing a Central Banker (16 points)
4. Portfolio choice (12 points)
5. Asset pricing (12 points)
6. *Ramsey model with taxes (6 points)*

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2. Precautionary Savings: Assume consumers maximize, $E_t\left[\sum_{t=0}^{\infty}\beta^t u(C_t)\right]$

s.t. $c_t + B_{t+1} + k_{t+1} = W_t$ **and** $W_t = \tilde{A}_t k_t^{\alpha} + (1+r)B_t$ *where* $\tilde{A}_t$ *is i.i.d.* $(\bar{A},\sigma^2)$

using the HARA utility function $U(c_t) = -\frac{1}{\phi}\exp(-\phi c_t)$

Where c_t and k_t are consumption and capital per capita, while B_t is a risk free asset paying certain return $R_t = (1 + r)$ and population growth n = 0.

a) Set up the Bellman equation for the consumer's maximization problem. Identify the state and control variables *(this answer was on the board if I recall).*

$$V(W_t) = u(c_t) + \beta E_t[V(W_{t+1})] \qquad (1.1)$$

$$W_{t+1} = \tilde{A}_{t+1}k_{t+1}^{\alpha} + R_t[W_t - c_t - k_{t+1}] \qquad (1.2)$$

The control variable is c_t and the state variables are k_{t+1} and A_t.

b) Is precautionary savings likely to be positive in this model, given that income from domestic capital is not diversifiable? Japelli and Pagano (1994) find OECD savings falls when credit markets develop further, arguing that this reflects lower precautionary savings? What is their reasoning, can this same argument explain the recent decline U.S. household savings?

Precaution savings will only be positive if the third derivative of the utility function is positive—we suspect it is because this is virtually the same utility function Blanchard and Fischer use in their chapter 7 discussion of precautionary savings (see chapter 6 page 288), but just in case, note that,

$$\frac{d}{dC}(-1/\phi)*\exp(-\phi * C) = e^{-\phi C}$$
$$\frac{d}{dC}(e^{-\phi C}) = -\phi e^{-\phi C}$$
$$\frac{d}{dC}(-\phi e^{-\phi C}) = \phi^2 e^{-C\phi}$$

Note that $u'''(C) = \phi^2 e^{-C\phi} > 0$ (see the discussion on page 354-57 of Romer 2nd ed.). Any utility function with a positive 3rd derivative leads to precautionary savings because if $u'''(C) > 0$ (the third derivative is positive) then marginal utility u'(C) is a convex function of C – see figure 7.3 (a great figure to draw for this answer). If future C_t is uncertain, the expected utility of future consumption is higher the more uncertain future income is (see figure 7.3b). Suppose in addition credit constrained households cannot borrow when incomes are low, the precautionary savings can be even more important (see Romer equation 7.42 for a discussion of the empirical magnitude of precautionary savings). However, if access to credit suddenly improves due to integration of international financial markets or interest rate liberalization (such as regulation q in the U.S.) for example, then precautionary savings may fall. Similarly an unexpected increase in wealth may reduce precautionary savings. Jappelli and Paganon (1994) test empirically whether greater access to credit reduces savings for OECD countries (see figure 7.4 on page 361 of Romer (2001)). They find higher savings rates in Japan and Spain can be explained by lower mortgage borrowing limits. Since home values have risen as home equity credit lines and mortgage markets have become much efficient in the U.S., this may be one explanation of why personal savings rates in the U.S. haven fallen over the past fifteen years (worsening the U.S. current account deficit). Greater access to credit reduces precautionary savings. More to point at hand, if more efficient credit markets move households in the direction predicted by intertemporal optimization models (built admittedly on some very strong assumption about household behavior) those models are validated to some extent (whereas a number of authors including Akerlof (2005) and Carroll and Weil (1996) argue habit and social norms are better explanations of household savings behavior across nations).

c) Derive the first order condition and sketch a strategy for solving the Bellman explicitly for the path on consumption recall the log normal distribution is,

$$E_t[\exp(x)] = \exp\left\{E(x) + \frac{\sigma_x^2}{2}\right\}$$

Hint: Use Adda and Cooper's direct attack method, (chapter 2) assuming a functional form for the value function and literally taking the expectation of $V(W_{t+1})$ – you may be able to make some progress using an envelope theorem, but this may be one of those problem where the envelope theorem is not helpful. This is more than a hint, as I was not able to solve this problem using the envelope theorem, if you found a way please send me your answer (typed or hand written). This solution strategy and utility function is similar to that used in Blanchard and Fischer's precautionary savings solution, albeit the derivation is a bit more complicated (see the discussion of this problem in Part 2 of the optimal consumption handout posted on the web page). To get full credit on this problem, you did not need to solve for the consumption path only lay out a solution strategy. But for completeness what follows is a complete solution. This problem is a simplified version of a 2003 MIT problem set question available online at MIT and here.

Recall since the exponential utility function is of the HARA class Merton (1971) shows that the value function has the same function form as the utility function—so much for the "art" of guessing the form of value functions. (HARA stands for hyperbolic risk aversion, see Blanchard and Fischer Chapter 6 pp. 283-84 and/or Professor Lengwiler's notes from the class handout – appended to this answer). In our case, this means the optimal value function will be of the general form,

$$V(W_t) = -\frac{1}{a\phi}\exp\left[-\phi(aW_t + b)\right] \tag{1.3}$$

$$V'(W_{t+1}) = \exp\left[-\phi(aW_{t+1} + b)\right] \tag{1.4}$$

All we have to do is solve for the undetermined coefficients a and b. Recalling from (1.2) and the that the given distribution of $\tilde{A}_t$ *is* $i.i.d.\,(\bar{A},\sigma^2)$, we have

$$E_t[W_{t+1}] = \bar{A}k_{t+1}^{\alpha} + (1+r)[W_t - c_t - k_{t+1}]$$
$$Var[W_{t+1}] = k_{t+1}^{2\alpha}\sigma^2 \quad (1.5)$$

Taking expectation of (1.3):

$$E_t[V(W_{t+1})] = E_t\left[-\frac{1}{a\phi}\exp\left[-\phi(aW_{t+1}+b)\right]\right]$$
$$E_t[V(W_{t+1})] = -\frac{1}{a\phi}\left[\exp\left[E_t(-\phi(aW_{t+1}+b)) + \frac{1}{2}Var(-\phi(aW_{t+1}+b))\right]\right] \quad \textbf{(1.6)}$$
$$E_t[V(W_{t+1})] = -\frac{1}{a\phi}\left[\exp\left[-\phi a[\bar{A}k_{t+1}^{\alpha} + (1+r)[W_t - c_t - k_{t+1}]] - \phi b + \frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2\right]\right]$$

We can write the Bellman equation using (1.6):

$$V(W_t) = -\frac{1}{\phi}\exp(-\phi c_t) - \frac{\beta}{a\phi}\left[\exp\left[-\phi a[\bar{A}k_{t+1}^{\alpha} + (1+r)[W_t - c_t - k_{t+1}]] - \phi b + \frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2\right]\right] \quad (1.7)$$

This implies the FOC for maximizing consumption given (1.7) is,

$$\exp(-\phi c_t) = \beta(1+r)\left[\exp\left[-\phi a[\bar{A}k_{t+1}^{\alpha} + (1+r)[W_t - c_t - k_{t+1}]] - \phi b + \frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2\right]\right] \quad (1.8)$$

Maximizing with respect to next periods wealth or k_{t+1}maximizes future consumption as well (see Adda and Cooper chapter 2):

$$[k_{t+1}:]\ 1+r = \alpha\bar{A}k_{t+1}^{\alpha-1} - \alpha\phi a k_{t+1}^{2\alpha-1}\sigma^2 \quad (1.9)$$

d) (more difficult) If you have time, solve explicitly for the time path of c_t and identify the term reflecting precautionary savings, that is "rainy day" savings in response to risk. Now we attempt to solve for a and b. To solve for the trajectory of c_t, i.e. the consumption function for this economy start by taking the log of (1.8) the consumption first order condition. This fairly common solution strategy yields,

$$-\phi c_t = \log\left[\beta(1+r)\right] - \phi a[\bar{A}k_{t+1}^{\alpha} + (1+r)[W_t - c_t - k_{t+1}]] - \phi b + \frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2$$

$$c_t = \frac{1}{\phi(1+a(1+r))}\left[-\log\left[\beta(1+r)\right] + \phi a[\bar{A}k_{t+1}^{\alpha} + (1+r)[W_t - k_{t+1}]] + \phi b - \frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2\right]$$

$$c_t = \frac{(1+r)a}{(1+a(1+r))}W_t + \frac{1}{\phi(1+a(1+r))}\left[-\log\left[\beta(1+r)\right] + \phi a[\bar{A}k_{t+1}^{\alpha} - (1+r)k_{t+1}] + \phi b - \frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2\right]$$

(1.10)

The first term is a fairly familiar looking potentially fixed "share of W_t" type consumption function, but the second term is of interest because it includes the variance of income which also affects consumption.

$$\varphi(a,b) = \frac{1}{\phi(1+a(1+r))}\left[-\log\left[\beta(1+r)\right] + \phi a[\bar{A}k_{t+1}^{\alpha} - (1+r)k_{t+1}] + \phi b - \frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2\right]$$

Therefore: **(1.11)**

$$c_t = \frac{(1+r)a}{(1+a(1+r))}W_t + \varphi(a,b)$$

At this point the solution is emerging: consumption is some fraction of current wealth W_t (the first term) plus some component which depends on the expectations of future income which is of course function of $k_{t+1.}$ The new kid on the block is the term which depends on the variance and enters with a negative sign. This is the precautionary savings term discussed in part (b) above: because the marginal utility of consumption is convex, greater anticipated variance of income – the last term in brackets—reduces current consumption out of current wealth thereby increasing savings, where as usual $c_t = (1-s)W_t$ where s is our yet to be determined savings rate. Since the only unknown term in this last expression is a, and it is squared, we can be fairly certain of our key result: increased variance of expected income raises the savings rate (this is precautionary savings).

For algebra lovers we can complete the solution by returning to (1.8) and making the simplify the assumption that, $\beta = \frac{1}{1+r}$ (both terms are going to be less than one so this is not an implausible assumption), this allows us to rewrite 1.8 as,

$$\exp(-\phi c_t) = \exp\left[-\phi a[\bar{A}k_{t+1}^{\alpha} + (1+r)[W_t - c_t - k_{t+1}]] - \phi b + \frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2\right] \quad (1.12)$$

Plugging this into the value function, $V(W_t) = u(c_t) + \beta E_t[W_{t+1}]$ yields,

$$-\frac{1}{a\phi}\exp\left[-\phi(aW_t+b)\right]=-\frac{1}{\phi}\exp(-\phi c_t)-\frac{\beta}{a\phi}\left[\exp\left[-\phi a[\bar{A}k_{t+1}^{\alpha}+(1+r)[W_t-c_t-k_{t+1}]]-\phi b+\frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2\right]\right]$$

$$-\frac{1}{a\phi}\exp\left[-\phi(aW_t+b)\right]=-\frac{1}{\phi}\exp(-\phi c_t)-\frac{\beta}{a\phi}\exp(-\phi c_t)$$

$$\frac{1}{a\phi}\exp\left[-\phi(aW_t+b)\right]=\left[\frac{a}{a\phi}+\frac{\beta}{a\phi}\right]\exp(-\phi c_t)$$

$$\frac{1}{a\phi}\exp\left[-\phi(aW_t+b)\right]=\left[\frac{a}{a\phi}+\frac{\beta}{a\phi}\right]\exp\left(-\phi\left[\frac{(1+r)a}{(1+a(1+r))}W_t+\varphi(a,b)\right]\right)$$

(1.13)

Collecting terms we have:

$$a=1-\beta \;\Rightarrow\; a=\frac{r}{1+r} \quad and \quad b=\varphi(a,b) \qquad (1.14)$$

Now lets return to our preliminary consumption function (1.11) and try these values:

$$c_t=\frac{r}{(1+r)}W_t+b=\frac{r}{(1+r)}W_t+b$$

$$\varphi(a,b)=\frac{1}{\phi(1+a(1+r))}\left[-\log\left[\beta(1+r)\right]+\phi a[\bar{A}k_{t+1}^{\alpha}-(1+r)k_{t+1}]+\phi b-\frac{1}{2}\phi^2 a^2 k_{t+1}^{2\alpha}\sigma^2\right]$$

$$b=\frac{1}{1+r}[\bar{A}k_{t+1}^{\alpha}-(1+r)k_{t+1}]-\frac{1}{2}\phi r\left(\frac{1}{1+r}\right)^2 k_{t+1}^{2\alpha}\sigma^2 \qquad \textbf{(1.15)}$$

Finally:

$$c_t=\frac{r}{(1+r)}W_t+\frac{1}{1+r}[\bar{A}k_{t+1}^{\alpha}-(1+r)k_{t+1}]-\frac{1}{2}\phi r\left(\frac{1}{1+r}\right)^2 k_{t+1}^{2\alpha}\sigma^2$$

Note that as expected, the last term is negative implying that a higher variance of expected consumption reduces current consumption and increases "precautionary savings." Compare this consumption function with that obtained by Blanchard and Fischer on page 290 of chapter 6.