Seeking the Slope of the Saddle Path in a 2x2 Phase Diagram

1. A system of two differential equations $\dot{y}(t) = Ay(t) + x(t)$ can be solved graphically or analytically, or numerically to illustrate the properties of the system. In all of these cases the following notation, calculations and properties may prove useful.

$$\begin{bmatrix} \dot{y}_1 \\ \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}_j * \begin{bmatrix} y_1 \\ \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} \quad \textit{or more compactly,} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. Stability: Even before actually computing the eigenvalues ω_1 and ω_2 (step 3) we can learn a lot about stability in the 2x2 case by just computing and signing **det A** = $\omega_1 * \omega_2$ and **tr A** = $\omega_1 + \omega_2$. If (Tr A)^2 > 4*(det A), ω_1 and ω_2 are real (imaginary roots are OK too, but we will not discuss these cases here). There are three possible outcomes:

Case 1: Unstable: **det A** = $f_{11} * f_{22} - f_{21} * f_{12}$ is positive; the roots must have the same sign.

So if trace of A (**tr A** = $\omega_1 + \omega_2$) is positive, the system is unstable

Case 2: Stable: Again if **det A** > 0 and the **tr A** is negative, the system is stable since both roots must be negative.

Case 3: Saddle path stability/instability: If **det A** is negative, the roots are real since the **(Tr A)^2 – 4*(det A)** is positive and they must be of different sign so the system saddle point stable. This is the case of interest here, as we seek the slope of the saddle path.

To summarize:

If $\omega_1 < 0, \omega_2 < 0$ **, the system is stable**

If $\omega_1 > 0, \omega_2 > 0$ **, the system is unstable**

If $\omega_1 < 0, \omega_2 > 0$ or $\omega_1 > 0, \omega_2 < 0$, the system is saddle path stable and the stable arm corresponds to the eigenvector associated with the negative eigenvalue.

3. The next step is to compute the eigenvalues. We set x_1, x_2 = 0 and work with a homogenous system, but see Barro and Sala-i-Martin p. 589 for a solution to the non-homogeneous system. We solve for the eigenvalues of **A**, ω_1 and ω_2 such tha**t,**

$$\det(A_j - \omega_i I) = \det \begin{vmatrix} f_{11} - \omega_i & f_{12} \\ f_{21} & f_{22} - \omega_i \end{vmatrix} = 0$$

Depending upon which is easiest to interpret, we can use the standard quadratic equation to solve for the roots, or more compact expression**,**

$$\omega_1, \omega_2 = \frac{(trA) \pm \sqrt{(trA)^2 - 4(\det A)}}{2}$$

4. Next we need to compute or derive the eigenvectors v_1, v_2, …,v_n such, that

$$(A_j - \omega I)v = 0$$

There are actually two sets of eigenvectors corresponding to the two eigenvalues ω_1, ω_2 Generally we only solve for the eigenvectors associated with the negative root, ω_2 for example (this is easier if we normalize the eigenvector setting the first element equal to 1 as in the system on the right).

$$\begin{bmatrix} f_{11} - \omega_2 & f_{12} \\ f_{21} & f_{22} - \omega_2 \end{bmatrix} \begin{bmatrix} v_{12}^{\omega_2} \\ v_{22}^{\omega_2} \end{bmatrix} = 0 \qquad \begin{bmatrix} f_{11} - \omega_i & f_{12} \\ f_{21} & f_{22} - \omega_i \end{bmatrix} \begin{bmatrix} 1 \\ v_{22}^{\omega_i} \end{bmatrix} = 0$$

5. Finally, it is time to "kill off" the explosive root and find the slope of the saddle path. Referring to Barro and Sala-Martin page 586, we have now solved our system of differential equations up to some arbitrary constant, so that, y = VEb where element v_{12} of the matrix V for example is the first element of the eigenvector associated with the second eigenvalue, so that,

$$y_1(t) = v_{11}e^{\omega_1 t}b_1 + v_{12}e^{\omega_2 t}b_2$$

$$y_2(t) = v_{21}e^{\omega_1 t}b_1 + v_{22}e^{\omega_2 t}b_2$$

Again if ω_2 is the negative root we can "kill" the influence of the unstable root by setting $b_1 = 0$. This is equivalent to picking an initial point on the stable arm, or on the saddle path. We now solve for $y_2 = f(y_1)$ by using 1st equation to solving for $e^{\omega_2 t}b_2$ substituting the result into the 2nd equation,

$$y_2(t) = \left[\frac{v_{22}}{v_{12}}\right] y_1(t) .$$

The term in brackets is the slope we seek: if y_1 is plotted on the X axis of the phase diagram is and y_2 plotted on the Y axis we have the slope of the saddle path in the neighborhood of the steady state (where the two phase lines cross). We're finished.